

Sum Formulas for Local Gromov-Witten Invariants of Spin Curves

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Abstract

Holomorphic 2-forms on Kähler surfaces lead to “Local Gromov-Witten invariants” of spin curves. This paper shows how to derive sum formulas for such local GW invariants from the sum formula for GW invariants of certain ruled surfaces. These sum formulas also verify the Maulik-Pandharipande formulas that were recently proved by Kiem and Li.

Let X be a Kähler surface with a holomorphic 2-form α . The real part of α , also denoted by α , then induces an almost complex structure on X :

$$J_\alpha = (Id + JK_\alpha)^{-1}J(Id + JK_\alpha). \quad (0.1)$$

Here J is the Kähler structure on X and K_α is the endomorphism of TX defined by the formula $\langle u, K_\alpha v \rangle = \alpha(u, v)$ where $\langle \cdot, \cdot \rangle$ is the Kähler metric on X . The almost complex structure J_α satisfies a remarkable *Image Localization Property*:

- if f is a J_α -holomorphic map into X that represents a non-zero $(1,1)$ class then the image of f lies in the zero set D of the holomorphic 2-form α

For simplicity, assume X is a (minimal) surface of general type and D is smooth. The normal bundle N to D is then a theta characteristic on D , i.e., N is a square root of the canonical bundle of D . The pair (D, N) is called a *spin curve* of genus h where h is the genus of D . The total space N_D of N has a tautological holomorphic 2-form α that induces, by the same manner as in (0.1), an almost complex structure J_α on N_D satisfying the image localization property, namely

$$\overline{\mathcal{M}}_{g,n}(N_D, d[D], J_\alpha) = \overline{\mathcal{M}}_{g,n}(D, d).$$

Consequently, the moduli space of J_α -holomorphic maps is compact, so it represents a (virtual) fundamental class that defines local GW invariants of the spin curve (D, N) . These local GW invariants depend only on the genus h and the parity $p \equiv h^0(N) \pmod{2}$ and GW invariants of Kähler surfaces with $p_g > 0$ are the sum of local GW invariants associated to spin curves [LP1].

GW invariants count maps from connected domains, while Gromov-Taubes invariants count maps from not necessarily connected domains. These GT invariants can be obtained from GW

invariants. Maulik and Pandharipande [MP] gave fascinating conjectural formulas for (descendent) local GT invariants of spin curve (D, N) for low degrees $(d = 1, 2)$:

$$\begin{aligned} GT_1^{loc, h, p} \left(\prod_{i=1}^n \tau_{k_i}(F^*) \right) &= (-1)^p \prod_{i=1}^n \frac{k_i!}{(2k_i + 1)!} (-2)^{-k_i} \\ GT_2^{loc, h, p} \left(\prod_{i=1}^n \tau_{k_i}(F^*) \right) &= (-1)^p 2^{h+n-1} \prod_{i=1}^n \frac{k_i!}{(2k_i + 1)!} (-2)^{k_i} \end{aligned} \tag{0.2}$$

(see Section 2 for definition of descendant local GT invariants). Kiem and Li [KL1, KL2, KL3] have since proved these formulas using their algebro-geometric version of local invariants. Observing the formulas (0.2) for genus zero spin curve directly follow from Proposition 2 of [FP], they showed the following reduction theorem:

- *for low degrees $(d = 1, 2)$ local GT invariants of higher genus spin curves can be reduced to local GT invariants of genus zero spin curve.*

Their proof uses a sum formula (Theorem 4.2 of [KL1]) for degeneration obtained by certain blow-up plus explicit calculation of the invariant $GT_2^{loc, h, p}(\tau(F^*))$.

The aim of this paper is twofold. First, we give a proof of the sum formula for degeneration by blow-up in the context of symplectic geometry — this sum formula is the same as Kiem and Li's sum formula except for the constraints of relative invariants (see Theorem A below). Second, we also prove new sum formulas for degeneration of spin curves in the case of degree $d = 2$ (see Theorem B below). Then, in the proof of the above reduction theorem (cf. Section 4 of [KL1]), we can replace the calculation of invariant $GT_2^{loc, h, p}(\tau)$ by Theorem B (see Section 9).

The novelty of our approach is to use GW invariants of ruled surfaces. Unlike the algebro-geometric approach, our local GW invariants of (D, N) are, in fact, *local contributions* to GW invariants of the ruled surface $\mathbb{P}_h = \mathbb{P}(N \oplus \mathcal{O}_D)$ that count maps whose images are close to the zero section D of \mathbb{P}_h . A small neighborhood U of D in \mathbb{P}_h is isomorphic to some neighborhood of the zero section in the total space N_D of N . Together with this isomorphism and some bump function, the tautological holomorphic 2-form on N_D induces an almost complex structure J_α on \mathbb{P}_h satisfying the image localization property, namely

$$\overline{\mathcal{M}}_{g,n}(U, dS, J_\alpha) = \overline{\mathcal{M}}_{g,n}(D, d)$$

where S is the section class of \mathbb{P}_h , i.e. $S = [D]$. The moduli space of J_α -holomorphic maps into U thus represents a (virtual) fundamental class that gives the local GW invariants of (D, N) . This description of local GW invariants is well suited to easily adapt the arguments in [IP1, IP2] to a version of sum formulas for local GW invariants. The relative local GW invariants are simply the local contributions to the relative GW invariants of \mathbb{P}_h that count maps into U relative a fixed fiber of \mathbb{P}_h . In terms of those relative invariants, sum formulas for local invariants directly follow from the main argument of [IP2] for some cases.

Our relative invariants are, however, not given by (virtual) fundamental class of relative moduli space that is needed to define descendent classes. To get around this issue, we relate descendent invariants to relative invariants with ϕ_i classes that are the first Chern classes of the

relative cotangent bundles over the space of stable curves (see Proposition 5.1). Then, we use those relative invariants to show the sum formula for degeneration by blow-up.

Let $\mathbb{F}_0 = \mathbb{P}^1 \times E$ be a ruled surface over $E = \mathbb{P}^1$. Then there is a unique section of \mathbb{F}_0 that passes through a given point. This simple observation enable us to apply the main argument of [IP2] for the symplectic fiber sum $\mathbb{P}_h = \mathbb{P}_h \#_V \mathbb{F}_0$ to obtain a sum formula for degeneration by blow-up. For any partition $m = (m_1, \dots, m_\ell)$, we set

$$\ell(m) = \ell, \quad |m| = \prod m_i, \quad m! = |\text{Aut}(m)|$$

where $\text{Aut}(m)$ is the symmetric group permuting equal parts of m . In Section 4, we show :

Theorem A. *Let $d \neq 0$ and $n_1 + n_2 = n$. Then*

$$\begin{aligned} & GT_d^{loc, h, p} \left(\prod_{i=1}^n \tau_{k_i}(F^*) \right) \\ &= \frac{1}{(d!)^2} \sum_m \frac{|m|}{m!} GT_{(1^d), m}^{loc, h, p} \left(\prod_{i=1}^{n_1} \phi_i^{k_i}(F^*) \right) \cdot GT_{m, (1^d)}^{\mathbb{F}_0} \left(\prod_{i=1}^{n_2} \phi_i^{k_{n_1+i}}(F^*) \right) \end{aligned} \quad (0.3)$$

where the sum is over all partitions m of d (see Section 3 for definition and notation of relative invariants).

Let $\mathbb{F}_1 = \mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}_E)$ be a ruled surface over E . Unlike the case of \mathbb{F}_0 , the infinite section plus a fiber represents the section class represented by the zero section of \mathbb{F}_1 . This causes the main difficulty to derive general sum formulas of local GW invariants for degeneration of spin curves from the symplectic fiber sums

$$\mathbb{P}_h = \mathbb{P}_{h_1} \#_{V_1} \mathbb{F}_1 \#_{V_2} \mathbb{P}_{h_2} \quad \text{and} \quad \mathbb{P}_h = \mathbb{P}_{h-1} \#_{V_1 \sqcup V_2} \mathbb{F}_1.$$

However, when degree $d = 2$, simple limiting arguments (see Section 6 and Lemma 7.3) allow us to apply the same arguments as in the proof of Theorem A. In Section 7 and 8, we show :

Theorem B.

(a) *If $h = h_1 + h_2$ and $p \equiv p_1 + p_2 \pmod{2}$ then we have*

$$GT_{(2)}^{loc, h, p} = (-1)^{p_1} 2^{h_1} GT_{(2)}^{loc, h_2, p_2} + (-1)^{p_2} 2^{h_2} GT_{(2)}^{loc, h_1, p_1} - (-1)^p 2^h GT_{(2)}^{loc, 0, +}. \quad (0.4)$$

(b) *If $h \geq 2$ or if $(h, p) = (1, +)$ then we have*

$$GT_{(2)}^{loc, h, p} = 4 GT_{(2)}^{loc, h-1, p} - (-1)^p 2^h GT_{(2)}^{loc, 0, +}. \quad (0.5)$$

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1 Moduli Spaces

This section introduces moduli spaces of curves and maps. For $2g + n \geq 3$, let $\overline{\mathcal{U}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}$ be the universal curve over the Deligne-Mumford space. Lifting to the moduli space of Prym curves ([Lo], [ACV]), one may assume that $\overline{\mathcal{M}}_{g,n}$ is a manifold and every connected n marked nodal curve C of (arithmetic) genus g has a stabilization $st(C) \in \overline{\mathcal{M}}_{g,n}$ that is isomorphic to a fiber of $\overline{\mathcal{U}}_{g,n}$. After fixing an embedding $\overline{\mathcal{U}}_{g,n} \hookrightarrow \mathbb{P}^N$, one can obtain a map

$$\phi : C \rightarrow st(C) \rightarrow \overline{\mathcal{U}}_{g,n} \rightarrow \mathbb{P}^N.$$

Let (X, ω) be a compact symplectic manifold with an ω -tamed almost complex structure J . A C^1 -map $f : C \rightarrow X$ is stable if the energy

$$E(f, \phi) = \frac{1}{2} \int |df|^2 + |d\phi|^2 \quad (1.1)$$

is positive on each (irreducible) component of C . An (irreducible) component of C is called a ghost component if the restriction f to that component represents a trivial homology class. Let ν be a section of the bundle $\text{Hom}(\pi_1^* T\mathbb{P}^N, \pi_2^* TX)$ over $\mathbb{P}^N \times X$ satisfying $J \circ \nu = -\nu \circ J_{\mathbb{P}^N}$. A stable map f is (J, ν) -holomorphic if

$$\frac{1}{2}(df + Jdfj) = (f, \phi)^* \nu$$

where j is the complex structure on C . Denote by $\overline{\mathcal{M}}_{g,n}(X, A)$ the moduli space of (J, ν) -holomorphic maps from nodal curves of (arithmetic) genus g with n marked points that represent the class A (we often omit (J, ν) in notation). We also denote by

$$\overline{\mathcal{M}}_{\chi,n}^*(X, A)$$

the moduli space of (J, ν) -holomorphic maps f from possibly disconnected domains of Euler characteristic χ with *no degree zero connected components*, namely the restriction of f to each “connected component” of its domain represents a non-trivial homology class.

For a finite set A , let $|A|$ denote the number of elements of A .

Remark 1.1. A stable map f in the moduli space $\overline{\mathcal{M}}_{\chi,n}^*(X, A)$ might have ghost components. Let $C = C_1 \cup C_2$ be the domain of f such that C_1 is a connected curve that is a union of some ghost components of f . Then, the stability of f implies that $2g(C_1) + \ell + n_1 \geq 3$ where $\ell = |C_1 \cap C_2|$ and n_1 is the number of marked points on C_1 .

The discussion below will be frequently used in subsequent arguments. Let ϕ_i be the first Chern class of line bundle over $\overline{\mathcal{M}}_{g,n}$ whose fiber over $(C, \{x_i\})$ is $T_{x_i}^* C$. For a subset I of $\{1, \dots, n\}$, let δ_I denote the Poincaré dual of the fundamental class of the boundary stratum of $\overline{\mathcal{M}}_{g,n}$ that consists of nodal curves $C_1 \cup C_2$ where C_1 has genus zero, C_2 has genus g and the marked points on C_1 are precisely those labeled by I .

Consider the forgetful map

$$\pi_k : \overline{\mathcal{M}}_{g,n+k} \rightarrow \overline{\mathcal{M}}_{g,n}$$

that forgets the last k marked points. For $1 \leq i \leq n$, we have

$$\pi_k^* \phi_i = \phi_i - \sum \delta_{\{i\} \cup I} \quad (1.2)$$

where the sum is over all $I \subset \{n+1, \dots, n+k\}$ with $I \neq \emptyset$ (cf. Lemma 3.1 of [AC]). The standard gluing map

$$\eta : \overline{\mathcal{M}}_{g_1, n_1+1} \times \overline{\mathcal{M}}_{g_2, n_2+1} \rightarrow \overline{\mathcal{M}}_{g_1+g_2, n_1+n_2} \quad (1.3)$$

is obtained by identifying the last marked point of the first component with the first marked point on the second component. For our purpose, we extend this gluing map to the cases where $2g_2 + n_2 < 2$. Denote by $\overline{\mathcal{M}}_{g,n}$ the space of one point when $2g + n < 3$ and note that

$$\overline{\mathcal{M}}_{g, n_1+1} \times \overline{\mathcal{M}}_{0,1} \cong \overline{\mathcal{M}}_{g, n_1} \quad \text{and} \quad \overline{\mathcal{M}}_{g, n_1+1} \times \overline{\mathcal{M}}_{0,2} \cong \overline{\mathcal{M}}_{g, n_1+1}.$$

Let η be the forgetful map that forgets the last marked point when $2g_1 + n_1 \geq 3$ and $(g_2, n_2) = (0, 0)$, and let η be the identity map when $(g_2, n_2) = (0, 1)$. The following fact then directly follows from (1.2) and Lemma 3.3 of [AC].

Lemma 1.2 ([AC]). *For $1 \leq i \leq n_1$ and $I \subset \{1, \dots, n_1\}$ with $2 \leq |I| < n_1$, we have*

(a) *if $2g_1 + n_1 \geq 3$ and $(g_2, n_2) = (0, 0)$ then*

$$\eta^* \phi_i = (\phi_i - \delta_{\{i, n_1+1\}}) \otimes 1 \quad \text{and} \quad \eta^* \delta_I = (\delta_I + \delta_{I \cup \{n_1+1\}}) \otimes 1$$

(b) *if either $(g_2, n_2) = (0, 1)$ or $2g_j + n_j \geq 2$ for $j = 1, 2$ then*

$$\eta^* \phi_i = \phi_i \otimes 1 \quad \text{and} \quad \eta^* \delta_I = \delta_I \otimes 1.$$

Lastly, we denote the space of curves with finitely many connected components, Euler class χ and n marked points by

$$\widetilde{\mathcal{M}}_{\chi, n}$$

(cf. page 57 of [IP1]). This space is a disjoint union of the products of the spaces $\overline{\mathcal{M}}_{g_j, n_j}$ with $\sum (2 - 2g_j) = \chi$ and $\sum n_j = n$ (including the unstable cases $2g_j + n_j < 3$). One can thus define ϕ_i classes and the boundary classes δ_I of $\widetilde{\mathcal{M}}_{\chi, n}$ in an obvious way.

2 Local GT Invariants

In this section, we introduce local Gromov-Taubes invariants of spin curves and set up notation for them. We will follow the definitions and notation in [RT2], [LT] and [IP1]. Let $\pi : N \rightarrow D$ be a theta characteristic on a smooth curve D of genus h . The canonical bundle of the total space N_D of N is then isomorphic to $\pi^* N$, so the tautological section of $\pi^* N$ gives a holomorphic 2-form α on N_D whose zero set is the zero section $D \subset N_D$ (cf. [LP1]). The projectivization

$$\mathbb{P}_h = \mathbb{P}(N \oplus \mathcal{O}_D)$$

is a ruled surface over D . For small $\epsilon > 0$ fix an isomorphism (denoted by Ψ) from a neighborhood of the zero section of \mathbb{P}_h to the 3ϵ -neighborhood of D in N_D taking the zero section of \mathbb{P}_h to

D . Choose a bump function β that is 1 on the ϵ -neighborhood of D in N_D and vanishes on the complement of 2ϵ -neighborhood of D . The pull-back 2-form $\Psi^*(\beta\alpha)$ is then a well-defined 2-form on \mathbb{P}_h . Fix a fiber V of the ruled surface \mathbb{P}_h and for small $\delta > 0$ choose a bump function β_V that is 1 on the complement of the 2δ -neighborhood of V and vanishes on δ -neighborhood of V . The 2-form

$$\alpha_V = \beta_V \Psi^*(\beta\alpha) \quad (2.1)$$

then induces, by (0.1), an almost complex structure J_V on \mathbb{P}_h . Let $U \subset \mathbb{P}_h$ be the preimage of ϵ -neighborhood of D in N_D under the isomorphism Ψ . We also denote by D the zero section of \mathbb{P}_h and let S be the section class represented by the zero section D of \mathbb{P}_h .

Lemma 2.1. *Let U and J_V be as above. Then every J_V -holomorphic map from a connected domain into \overline{U} that represents the class dS with $d \neq 0$ is, in fact, holomorphic and its image lies entirely in D , i.e.*

$$\overline{\mathcal{M}}_{\chi,n}^*(\overline{U}, dS, J_V) = \overline{\mathcal{M}}_{\chi,n}^*(D, d).$$

Proof. This proof is similar to that of Lemma 3.2 in [LP1]. Use the same notation α for the real part of the holomorphic 2-form α on N_D . Let $f : (C, j) \rightarrow \overline{U}$ be a J_V -holomorphic map from a connected curve C with complex structure j that represents the class dS where $d \neq 0$. For each point $q \in C$, let $\{e_1, e_2 = je_1\}$ be an orthonormal basis of $T_q C$. Then

$$|\overline{\partial}f|^2 = f^*|\beta_V \Psi^*(\beta\alpha)|^2 |\partial f|^2 = f^*(\beta_V \Psi^*(\beta\alpha))(e_1, e_2) \leq f^*(\Psi^*\alpha)(e_1, e_2) \quad (2.2)$$

where the two equalities follow from Proposition 1.3 of [L] and the inequality follows from the facts (i) $\Psi^*\beta \equiv 1$ on \overline{U} and (ii) $0 \leq \beta_V \leq 1$. Since α is a real part of holomorphic 2-form, integrating over the domain shows f is indeed holomorphic and the image of f lies in the zero set of $\beta_V \Psi^*(\beta\alpha)$ in \overline{U} . Since f represents the class dS , the image of f must lie in the zero set of $\Psi^*(\beta\alpha)$ in \overline{U} which is the zero section D . This completes the proof. \square

Remark 2.2. Let α be a holomorphic 2-form on U and g be any function on U that satisfies $0 \leq g \leq 1$. Then, the inequality in (2.2) shows that for $\tilde{\alpha} = g\alpha$ every $J_{\tilde{\alpha}}$ -holomorphic map into U representing the class dS ($d \neq 0$) is holomorphic and has its image lying in the zero set of $\tilde{\alpha}$.

Let $d \neq 0$ and fix (J, ν) that is close to $(J_V, 0)$. Lemma 2.1 and the Gromov Compactness Theorem imply that the moduli space of (J, ν) -holomorphic maps $\overline{\mathcal{M}}_{\chi,n}^*(U, dS)$ is compact. The construction of Li and Tian [LT] then defines the (virtual) fundamental class

$$[\overline{\mathcal{M}}_{\chi,n}^*(U, dS)]^{vir} \in H_*(\mathcal{M}ap_{\chi,n}(\mathbb{P}_h, dS); \mathbb{Q}) \quad (2.3)$$

in the homology of the space $\mathcal{M}ap_{\chi,n}(\mathbb{P}_h, dS)$ of stable maps into \mathbb{P}_h from nodal curves of Euler characteristic χ with n marked points that represent the homology class dS .

Definition 2.3. *The local (descendent) GT invariants of the spin curve (D, N) of genus h with parity $p \equiv h^0(N) \pmod{2}$ are :*

$$GT_d^{loc, h, p} \left(\prod_{i=1}^n \tau_{k_i}(F^*) \right) = [\overline{\mathcal{M}}_{\chi,n}^*(U, dS)]^{vir} \cap \left(\prod_{i=1}^n \psi_i^{k_i} \cup ev_i^*(F^*) \right) \quad (2.4)$$

where ψ_i be the Euler class of the bundle over $\mathcal{M}ap_{\chi,n}(\mathbb{P}_h, dS)$ whose fiber over $(f, C, \{x_i\})$ is $T_{x_i}^*C$, F^* is the Poincaré dual of the fiber class of \mathbb{P}_h , ev_i is the evaluation map at the i -th marked point, and the Euler characteristic χ satisfies

$$\sum k_i = d(1-h) - \frac{1}{2}\chi.$$

We will often write respectively $+$ and $-$ for $p \equiv 0 \pmod{2}$ and for $p \equiv 1 \pmod{2}$.

Remark 2.4. Let β_t be a path from $\beta_0 = \beta_V$ to $\beta_1 \equiv 1$ with $0 \leq \beta_t \leq 1$ on \mathbb{P}_h and let J_t denote the almost complex structure induced from the 2-from $\beta_t \Psi^*(\beta\alpha)$ by (0.1). The proof of Lemma 2.1 shows for $d \neq 0$ and for all t

$$\overline{\mathcal{M}}_{\chi,n}^*(\overline{U}, dS, J_t) = \overline{\mathcal{M}}_{\chi,n}^*(D, d).$$

In particular, this shows $\overline{\mathcal{M}}_{\chi,n}^*(\overline{U}, dS, J_t)$ is compact for all t . It then follows from the standard cobordism argument (cf. Proposition 2.3 of [LT]) that the (virtual) fundamental class (2.3) is independent of the choice of J_t . So, when $t = 1$, the isomorphism Ψ as above gives

$$[\overline{\mathcal{M}}_{\chi,n}^*(U, dS)]^{vir} \cap \left(\prod_{i=1}^n \psi_i^{k_i} \cup ev_i^*(F^*) \right) = [\overline{\mathcal{M}}_{\chi,n}^*(N_D, d[D])]^{vir} \cap \left(\prod_{i=1}^n \psi_i^{k_i} \cup ev_i^*(\pi^* \gamma^*) \right)$$

where $\gamma^* \in H^2(D)$ is Poincaré dual of the point class of D . Thus, given χ, n and $d \neq 0$ the invariant (2.4) depends only on the genus h of D and the parity $p = h^0(N) \pmod{2}$.

The stabilization and evaluation at marked points defines a map

$$\varepsilon = st \times ev : \mathcal{M}ap_{\chi,n}(\mathbb{P}_h, dS) \rightarrow \widetilde{\mathcal{M}}_{\chi,n} \times (\mathbb{P}_h)^n. \quad (2.5)$$

For the classes ϕ_i on $\widetilde{\mathcal{M}}_{\chi,n}$, we set

$$GT_d^{loc,h,p} \left(\prod_{i=1}^n \phi_i^{k_i}(F^*) \right) = [\overline{\mathcal{M}}_{\chi,n}^*(U, dS)]^{vir} \cap \left(\prod_{i=1}^n st^* \phi_i^{k_i} \cup ev_i^*(F^*) \right).$$

Now, suppose (J, ν) is generic (see page 10 of [RT2]). Then, the image of $\overline{\mathcal{M}}_{\chi,n}^*(U, dS)$ under the map (2.5) defines a homology class

$$[\overline{\mathcal{M}}_{\chi,n}^*(U, dS)] \in H_* \left(\widetilde{\mathcal{M}}_{\chi,n} \times (\mathbb{P}_h)^n; \mathbb{Q} \right) \quad (2.6)$$

satisfying $\varepsilon_* [\overline{\mathcal{M}}_{\chi,n}^*(U, dS)]^{vir} = [\overline{\mathcal{M}}_{\chi,n}^*(U, dS)]$ (cf. Remark 10.2 of [LP2]). So, we have

$$GT_d^{loc,h,p} \left(\prod_{i=1}^n \phi_i^{k_i}(F^*) \right) = [\overline{\mathcal{M}}_{\chi,n}^*(U, dS)] \cap \prod_{i=1}^n \phi_i^{k_i} \otimes (F^*)^{\otimes n}.$$

Remark 2.5. If the spin curve (D, N) has genus $h > 0$, then there are no non-constant holomorphic maps from genus zero curves to D and hence, by the Gromov Compactness Theorem and Lemma 2.1, for every map f in the moduli space $\overline{\mathcal{M}}_{\chi, n}^*(U, dS)$ every genus zero (irreducible) component is ghost component. It thus follows from the stability and the relation between ψ_i class and $st^*\phi_i$ class (cf. [KM] page 388) that

$$GT_d^{loc, h, p} \left(\prod_{i=1}^n \tau_{k_i}(F^*) \right) = GT_d^{loc, h, p} \left(\prod_{i=1}^n \phi_i^{k_i}(F^*) \right). \quad (2.7)$$

We end this section with dimension zero local GT invariants for $d = 1$ and 2 .

Lemma 2.6. $GT_1^{loc, h, p} = (-1)^p$ and $GT_2^{loc, h, p} = (-1)^p 2^{h-1}$.

Proof. The dimension zero local GW invariants $GW_d^{loc, h, p}$ and the dimension zero local GT invariants $GT_d^{loc, h, p}$ are related as follows:

$$1 + \sum_{d>0} GT_d^{loc, h, p} t^d = \exp \left(\sum_{d>0} GW_d^{loc, h, p} t^d \right)$$

(cf. Section 2 of [IP2]). The lemma thus follows from the fact

$$GW_1^{loc, h, p} = (-1)^p \quad \text{and} \quad GW_2^{loc, h, p} = \frac{1}{2} [(-1)^p 2^h - 1]$$

(see Section 10 of [LP1]). \square

3 Relative Local Invariants

In [IP1], GW invariants were generalized to relative GW invariants relative to codimension two symplectic submanifold. Following [IP1], we can define relative local invariants. A (J, ν) -holomorphic map f is called *V-regular with a contact vector* $s = (s_1, \dots, s_\ell)$ if $f^{-1}(V)$ consists of the last ℓ ordered marked points $x_{n+1}, \dots, x_{n+\ell}$ such that the image of f has the contact order s_k at x_{n+k} . Denote by

$$\mathcal{M}_{\chi, n, s}^{V, *}(U, dS)$$

the moduli space of V -regular (J, ν) -holomorphic maps f into U with contact vector s where the superscript $*$ also means f has no degree zero connected components. For a contact vector $s = (s_1, \dots, s_\ell)$, we write

$$\deg(s) = \sum_{k=1}^{\ell} s_k, \quad \ell(s) = \ell, \quad |s| = \prod_{k=1}^{\ell} s_k$$

and, noting there are no rim tori since $H_1(V) = 0$ (cf. Remark 5.3 of [IP1]), we set

$$V_s = \{ (v_1, s_1), \dots, (v_\ell, s_\ell) \mid v_k \in V \}.$$

The moduli space of V -regular maps $\mathcal{M}_{\chi,n,s}^{V,*}(U, dS)$ also has an associated map

$$\varepsilon_s = st \times ev \times h_s : \mathcal{M}_{\chi,n}^{V,*}(U, dS) \rightarrow \widetilde{\mathcal{M}}_{\chi,n+\ell(s)} \times (\mathbb{P}_h)^n \times V_s \quad (3.1)$$

where ev is the evaluation map at the first n marked points and h_s is given by

$$h_s(f, x_1, \dots, x_{n+\ell}) = ((f(x_{n+1}), s_1), \dots, (f(x_{n+\ell}), s_\ell)). \quad (3.2)$$

Observe that the (holomorphic) fiber V of \mathbb{P}_h is J_V -holomorphic since the 2-form α_V in (2.1) vanishes near V . The pair $(J_V, 0)$ is thus V -compatible in the sense of Definition 3.2 of [IP1]. Now, choose a generic V -compatible (J, ν) that is sufficiently close to $(J_V, 0)$. Lemma 2.1, the Gromov Compactness Theorem and the relative GW theory of [IP1] then imply that the image of the moduli space $\mathcal{M}_{\chi,n,s}^{V,*}(U, dS)$ under the map (3.1) defines a homology class

$$[\mathcal{M}_{\chi,n,s}^{V,*}(U, dS)] \in H_*(\widetilde{\mathcal{M}}_{\chi,n+\ell(s)} \times (\mathbb{P}_h)^n \times V_s; \mathbb{Q}) \quad (3.3)$$

Let $\{\beta_j\}$ be a basis of $H_*(V)$. Then a basis of $H^*(V_s)$ is given by elements of the form

$$C_{J,s} = C_{\beta_{j_1}, s_1} \otimes \dots \otimes C_{\beta_{j_\ell}, s_\ell}.$$

Lemma 2.1 and the Gromov Compactness Theorem imply that

$$[\mathcal{M}_{\chi,n,s}^{V,*}(U, dS)] \cap C_{J,s} = 0 \quad (3.4)$$

unless all β_{j_k} are the fundamental class $[V]$ of V . Since $H_1(V) = 0$, we can forget the ordering of the contact constraints $C_{J,s}$ by simply writing

$$\prod_{j,b} (C_{\beta_j,b})^{m_{j,b}} = C_{\beta_{j_1}, s_1} \dots C_{\beta_{j_\ell}, s_\ell} \quad (3.5)$$

with the relation $C_{\beta_j,b} \cdot C_{\beta_i,a} = C_{\beta_i,a} \cdot C_{\beta_j,b}$ where $m = (m_{j,b})$ is a sequence of nonnegative integers determined by (3.5). If all $\beta_{j_k} = \beta_j$ for some j then the sequence m can be considered as a partition of the integer d . The (unordered) contact constraint (3.5) is then a pair of the Poincaré dual of β_j and the partition m of d , i.e. $m = (m_1, \dots, m_\ell)$ with $m_1 \leq m_2 \leq \dots \leq m_\ell$ and $\sum m_j = d$. Write $m = (1^d)$ if all $m_j = 1$. In that case,

$$|m| = |(1^d)| = 1 \quad \text{and} \quad m! = (1^d)! = d!.$$

When all $\beta_{j_k} = \beta_j$ for some j , we write $C_{J,s}$ simply as $C_{\beta_j^\ell}$.

Remark 3.1. Given a partition $m = (m_1, \dots, m_\ell)$ of d , there are $\ell!/m!$ ordered sequences s with $\tau(s) = m$ for some permutation τ in the symmetric group S_ℓ .

Definition 3.2. For a partition m of d with $m = \tau(s)$ for some permutation τ in $S_{\ell(s)}$, we set

$$GT_m^{loc,h,p} \left(\prod_{i=1}^n \phi_i^{k_i}(F^*) \right) = [\mathcal{M}_{\chi,n,s}^{V,*}(U, dS)] \cap \prod_{i=1}^n \phi_i^{k_i} \otimes (F^*)^n \otimes C_{[V]^{\ell(m)}}$$

where the Euler characteristic χ is given by

$$\sum k_i = d(1-h) - \frac{1}{2}\chi + (\ell(m) - d).$$

Choose two distinct fibers V_1 and V_2 of \mathbb{P}_h and let $V = V_1 \sqcup V_2$. We can also define an almost complex structure J_V that equals to the Kähler structure of \mathbb{P}_h near V and satisfies Lemma 2.1. Thus, we can define relative local invariants

$$GT_{m^1, m^2}^{loc, h, p} \left(\prod_{i=1}^n \phi_i^{k_i}(F^*) \right) \quad (3.6)$$

relative to V (with contact vectors m^i with V_i) for the class dS with Euler characteristic χ where the Euler characteristic χ satisfies

$$\sum k_i = d(1-h) - \frac{1}{2}\chi + \sum (\ell(m^i) - d).$$

Remark 3.3. The only genus zero spin curve is the even spin curve $(\mathbb{P}^1, \mathcal{O}(-1))$. In this case, $S^2 = -1$ and hence for $d \neq 0$ we have

$$\overline{\mathcal{M}}_{\chi, n}^*(U, dS, J_V) = \overline{\mathcal{M}}_{\chi, n}^*(\mathbb{P}_0, dS, J_V).$$

This shows that degree d local invariants of spin curve $(\mathbb{P}^1, \mathcal{O}(-1))$ are the same as the GW invariants of \mathbb{P}_0 for the class dS . It also shows that relative local invariants are equal to the relative GW invariants of (\mathbb{P}_0, V) .

For simplicity, we set

$$\overline{\mathcal{M}} = \overline{\mathcal{M}}_{\chi, n}^*(U, dS) \quad \text{and} \quad \mathcal{M}^V = \mathcal{M}_{\chi, n, m^1, m^2}^{V_1, V_2, *} (U, dS).$$

Noting the homology class (2.6) defines a map $H^*((\mathbb{P}_h)^n) \rightarrow H_*(\widetilde{\mathcal{M}}_{\chi, n})$, we set

$$GT_{d, \chi}^{loc, h, p}((F^*)^n) = [\overline{\mathcal{M}}] \cap (F^*)^{\otimes n} \in H_{2q}(\widetilde{\mathcal{M}}_{\chi, n})$$

where $q = d(1-h) - \frac{1}{2}\chi$. Similarly, we also set

$$GT_{m^1, m^2, \chi}^{loc, h, p}((F^*)^n) = [\mathcal{M}^V] \cap (F^*)^n \bigotimes_{i=1}^2 C_{[V_i] \ell(m^i)} \in H_{2r}(\widetilde{\mathcal{M}}_{\chi, n + \sum \ell(m^i)})$$

where $r = d(1-h) - \frac{1}{2}\chi + \sum (\ell(m^i) - d)$.

Remark 3.4. Let B be a geometric representative of the n product of fiber classes $F^{\otimes n}$ of $(\mathbb{P}_h)^n$ in general position with respect to the evaluation map at marked points. Then the images of the cut-down moduli spaces $\overline{\mathcal{M}} \cap B$ and $\mathcal{M}^V \cap B$ under the stabilization map respectively define classes satisfying

$$[st(\overline{\mathcal{M}} \cap B)] = GT_{d, \chi}^{loc, h, p}((F^*)^n) \quad \text{and} \quad [st(\mathcal{M}^V \cap B)] = GT_{m^1, m^2, \chi}^{loc, h, p}((F^*)^n).$$

For the ruled surface $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$, we use the same notations F and S for the fiber class and the section class, respectively. To save notation, we also use the same notation V for the union of 2 distinct fibers V_1 and V_2 of the ruled surface \mathbb{F}_0 . For partitions m^i of d , denote by

$$GT_{m^1, m^2}^{\mathbb{F}_0} \left(\prod_{i=1}^n \phi_i^{k_i}(F^*) \right) \quad (3.7)$$

the relative GT invariants of (\mathbb{F}_0, V) for the class dS with Euler characteristic χ satisfying

$$\sum k_i = d - \frac{1}{2}\chi + (\ell(m^2) - d)$$

where the contact constraint with V_1 and V_2 are respectively $C_{pt^{\ell(m^1)}}$ and $C_{[V_2]^{\ell(m^2)}}$. It follows directly from Lemma 14.6 of [IP2] that

$$GT_{(1^d)}^{\mathbb{F}_0} = 1 \quad \text{and} \quad GT_{(1^d), (1^d)}^{\mathbb{F}_0} = d! \quad (3.8)$$

For the class $[\mathcal{M}_{\chi, n}^V(\mathbb{F}_0, dS)]$ that defines the relative invariants (3.7) of (\mathbb{F}_0, V) we also set

$$GT_{m^1, m^2, \chi}^{\mathbb{F}_0}((F^*)^n) = [\mathcal{M}_{\chi, n}^V(\mathbb{F}_0, dS)] \cap (F^*)^n \otimes C_{pt^{\ell(m^1)}} \otimes C_{[V_2]^{\ell(m^2)}}.$$

This is a homology class in $H_{2t}(\widetilde{\mathcal{M}}_{\chi, n + \sum \ell(m^i)})$ where $t = d - \frac{1}{2}\chi + \sum(\ell(m^2) - d)$.

4 Blow-Up and Sum Formula

The aim of this section is to prove Theorem A in the Introduction. We will apply the limiting and smoothing arguments of [IP2] to our local invariants. The proof consists of three steps.

Step 1 : Fix a fiber V_0 of \mathbb{P}_h and consider a degeneration

$$\lambda : Z \xrightarrow{\sigma} \mathbb{P}_h \times \mathbb{C} \longrightarrow \mathbb{C}$$

where $\sigma : Z \rightarrow \mathbb{P}_h \times \mathbb{C}$ is the blow-up of $\mathbb{P}_h \times \mathbb{C}$ along $V_0 \times \{0\}$ and the second map is projection onto the second factor. The central fiber Z_0 is the singular surface $\mathbb{P}_h \cup_{V_0} \mathbb{F}_0$ and general fibers Z_λ ($\lambda \neq 0$) are isomorphic to \mathbb{P}_h that is the symplectic fiber sum $\mathbb{P}_h \#_{V_0} \mathbb{F}_0$. For $\lambda \neq 0$, let D_λ denote the zero section of Z_λ , i.e., $D_\lambda = \sigma^{-1}(D \times \{\lambda\})$.

Choose a fiber $V_1 \neq V_0$ of $\mathbb{P}_h \subset Z_0$ and a fiber $V_2 \neq V_0$ of $\mathbb{F}_0 \subset Z_0$ and set

$$V = V_1 \sqcup V_0 \sqcup V_2.$$

One can choose a (smooth) family \tilde{V}_λ of disjoint union of two fibers of $Z_\lambda \simeq \mathbb{P}_h$ with $\tilde{V}_0 = V_1 \sqcup V_2$. Denote by $\mathcal{J}(Z)$ the space of all (J, ν) on Z satisfying (i) each Z_λ is J -invariant and (ii) the restriction of (J, ν) to Z_λ ($\lambda \neq 0$) is \tilde{V}_λ -compatible and to Z_0 is V -compatible (cf. Lemma 2.3 of [IP2]). We will use the same notation (J, ν) for its restriction to each Z_λ .

Fix a small $\delta > 0$ and define a δ -neck $Z(\delta)$ as a (normal) δ -neighborhood of V in Z . The energy of a map f (more precisely of (f, ϕ) as in (1.1)) into Z in the δ -neck is

$$E^\delta(f) = \frac{1}{2} \int |df|^2 + |d\phi|^2$$

where the integral is over $f^{-1}(Z(\delta))$. By Lemma 1.5 of [IP1] there is a constant c_V depending only on the restriction of (J, ν) to $V \subset Z$ such that every component of every (J, ν) -holomorphic map into V has energy greater than c_V . A (J, ν) -holomorphic map f into Z is δ -flat if the energy

in the δ -neck $E^\delta(f)$ is at most $c_V/2$. Note that a δ -flat map into Z_0 has no components mapped entirely into V .

Once and for all, fix $\chi, n, d \neq 0$ and for each $\lambda \neq 0$ we set

$$\mathcal{M}(Z_\lambda) = \mathcal{M}_{\chi, n, (1^d), (1^d)}^{\tilde{V}_\lambda}(Z_\lambda, dS).$$

Denote the set of δ -flat maps in $\mathcal{M}(Z_\lambda)$ by $\mathcal{M}^\delta(Z_\lambda)$ and write

$$\lim_{\lambda \rightarrow 0} \mathcal{M}^\delta(Z_\lambda) \quad (4.1)$$

for the set of limits of sequences of δ -flat maps in $\mathcal{M}^\delta(Z_\lambda)$ as $\lambda \rightarrow 0$. Since δ -flatness is a closed condition, each map f in the limit set (4.1) is also δ -flat and hence the domain of f has no components mapped entirely into V . Consequently, we have

- (a) f splits as $f = (f_1, f_2)$ where f_1 and f_2 are respectively $(V_1 \sqcup V_0)$ -regular map into \mathbb{P}_h and $(V_0 \sqcup V_2)$ -regular map into \mathbb{F}_0 and each f_i has contact vector (1^d) with V_i for $i = 1, 2$,
- (b) $f^{-1}(V_0)$ consists of nodes $\{p_1, \dots, p_\ell\}$ of the domain such that each p_i has a well-defined multiplicity s_i equal to the order of contact of the image of f_1 (or f_2) with V_0 at p_i

(see Section 3 of [IP2]). Renumbering the nodes $\{p_1, \dots, p_\ell\}$ gives $\ell!$ ordered sequences $s = (s_1, s_2, \dots, s_\ell)$. On the other hand, since for small $|\lambda|$ the δ -flat maps in $\mathcal{M}^\delta(Z_\lambda)$ are C^0 -close to δ -flat maps in the limit set (4.1), to each map f in $\mathcal{M}^\delta(Z_\lambda)$ one can assign ordered sequences s . Denote by $\mathcal{M}_s^\delta(Z_\lambda)$ the set of all such pair (f, s) labeled by an ordered sequence s . Then, there are actions of symmetric groups S_ℓ such that

$$\bigsqcup_{\ell} \left(\bigsqcup_{\ell(s)=\ell} \mathcal{M}_s^\delta(Z_\lambda) \right) / S_\ell = \mathcal{M}^\delta(Z_\lambda). \quad (4.2)$$

For $\mathbb{P}_h, \mathbb{F}_0 \subset Z_0$ and each ordered sequence s with $\deg(s) = d$, there is an evaluation map

$$ev_s : \bigcup \left(\mathcal{M}_{\chi_1, n_1, (1^d), s}^{V_1, V_0}(\mathbb{P}_h, dS - kF) \times \mathcal{M}_{\chi_2, n_2, s, (1^d)}^{V_0, V_2}(\mathbb{F}_0, dS + kF) \right) \longrightarrow V_0^{\ell(s)} \times V_0^{\ell(s)} \quad (4.3)$$

that records the intersection points with the fiber V_0 where the union is over all $0 \leq k \leq h$, $n_1 + n_2 = n$ and $\chi = \chi_1 + \chi_2 - 2\ell(s)$. Let Δ_s be the diagonal of $V_0^{\ell(s)} \times V_0^{\ell(s)}$ and denote by

$$\mathcal{K}_s^\delta \subset ev_s^{-1}(\Delta_s)$$

the set of δ -flat maps in $ev_s^{-1}(\Delta_s)$. Since each map in $ev_s^{-1}(\Delta_s)$ can be considered as a pair of a map f into $Z_0 = \mathbb{P}_h \cup_{V_0} \mathbb{F}_0$ satisfying (a) and (b) with an ordered sequence s , we have

$$\lim_{\lambda \rightarrow 0} \bigsqcup_s \mathcal{M}_s^\delta(Z_\lambda) \subset \bigsqcup_s \mathcal{K}_s^\delta. \quad (4.4)$$

Conversely, each map $f = (f_1, f_2) \in \mathcal{K}_s^\delta$ can be smoothed to produce maps in $\mathcal{M}_s^\delta(Z_\lambda)$ for small $|\lambda|$. Let C_1 and C_2 be the domains of f_1 and f_2 respectively. Identifying the $\ell(s)$ contact points with V_0 of C_1 with the $\ell(s)$ contact points with V_0 of C_2 determines a gluing map

$$\widetilde{\mathcal{M}}_{\chi_1, n_1 + \ell(s) + d} \times \widetilde{\mathcal{M}}_{\chi_2, n_2 + \ell(s) + d} \longrightarrow \widetilde{\mathcal{M}}_{\chi, n + 2d}$$

where $\chi_i = \chi(C_i)$ for $i = 1, 2$. For each $\ell(s)$, taking the union over all χ_1, χ_2, n_1 and n_2 defines a gluing map

$$\xi_{\ell(s)} : \bigsqcup \widetilde{\mathcal{M}}_{\chi_1, n_1 + \ell(s) + d} \times \widetilde{\mathcal{M}}_{\chi_2, n_2 + \ell(s) + d} \longrightarrow \widetilde{\mathcal{M}}_{\chi, n + 2d}$$

Theorem 10.1 of [IP2] then gives :

Theorem 4.1 ([IP2]). *For generic $(J, \nu) \in \mathcal{J}(Z)$ and for small $|\lambda|$, there is an $|s|$ -fold covering*

$$\pi_{s, \lambda} : \mathcal{K}_s^\delta(Z_\lambda) \rightarrow \mathcal{K}_s^\delta$$

with a commutative diagram (up to homotopy) :

$$\begin{array}{ccc} \bigsqcup_s \mathcal{K}_s^\delta(Z_\lambda) & \xrightarrow{\Phi_\lambda} & \bigsqcup_s \mathcal{M}_s^\delta(Z_\lambda) \\ \downarrow st_\lambda & & \downarrow st \\ \bigsqcup_s \widetilde{\mathcal{M}}_{\chi_1, n_1 + \ell(s) + d} \times \widetilde{\mathcal{M}}_{\chi_2, n_2 + \ell(s) + d} & \xrightarrow{\xi} & \widetilde{\mathcal{M}}_{\chi, n + 2d} \end{array} \quad (4.5)$$

where the top arrow is an embedding, $st_\lambda = st \circ \pi_{s, \lambda}$ and ξ in the bottom arrow is given by the gluing maps $\xi_{\ell(s)}$. The construction of the smoothing map Φ_λ also shows that

$$\lim_{\lambda \rightarrow 0} \Phi_\lambda(\tilde{f}_\lambda) = f \quad \text{where} \quad \pi_{s, \lambda}(\tilde{f}_\lambda) = f. \quad (4.6)$$

Step 2 : Let U be an open neighborhood of the zero section D of \mathbb{P}_h and $\alpha_0 = \alpha_{V_0}$ be a 2-form as in Section 2 (see the paragraph above Lemma 2.1). Regard the 2-form α_0 as a 2-form on $\mathbb{P}_h \times \mathbb{C}$ in an obvious way and for small $\epsilon > 0$ choose a bump function β that is 1 on the complement of 2ϵ -neighborhood of V_1 in Z and vanishes on ϵ -neighborhood of V_1 in Z . The 2-form $\beta(p^*\alpha_0)$ then defines, again by the formula (0.1), an almost complex structure J_V on Z . The pair $(J_V, 0) \in \mathcal{J}(Z)$ and the restriction of J_V to \mathbb{F}_0 is the product complex structure of \mathbb{F}_0 since $p^*\alpha_0$ vanishes on some neighborhood of \mathbb{F}_0 in Z . Moreover, by Remark 2.2, we have

$$\begin{aligned} \overline{\mathcal{M}}_{\chi, n + 2d}^*(U_\lambda, dS, J_V) &= \overline{\mathcal{M}}_{\chi, n + 2d}^*(D_\lambda, d) \\ \overline{\mathcal{M}}_{\chi, n + 2d}^*(U_0 \cap \mathbb{P}_h, dS, J_V) &= \overline{\mathcal{M}}_{\chi, n + 2d}^*(D, d) \end{aligned} \quad (4.7)$$

where $U_\lambda = p^{-1}(U \times \{\lambda\})$. Fix an open neighborhood W of D in \mathbb{P}_h satisfying $\overline{W} \subset U$ and for each λ set

$$W_\lambda = p^{-1}(W \times \{\lambda\})$$

The following fact is our key observation for the proof of Theorem A.

Lemma 4.2. *For $(J, \nu) \in \mathcal{J}(Z)$ sufficiently close to $(J_V, 0)$ and for small $|\lambda| > 0$, we have*

$$\overline{\mathcal{M}}_{\chi, n + 2d}^*(U_\lambda, dS) \setminus \overline{\mathcal{M}}_{\chi, n + 2d}^*(W_\lambda, dS) = \emptyset.$$

Proof. Suppose not. Then there exists a sequence of (J_k, ν_k) -holomorphic maps f_k into U_{λ_k} with $\text{Im}(f_k) \cap (U_{\lambda_k} \setminus W_{\lambda_k}) \neq \emptyset$ and with no degree zero connected components where λ_k converges to 0 and (J_k, ν_k) converges to $(J_V, 0)$ as $k \rightarrow \infty$. The Gromov Compactness Theorem then implies that after passing to subsequences, f_k converges to a J_V -holomorphic map f into Z_0 such that

(i) $\text{Im}(f) \subset \overline{U}_0$ and (ii) $\text{Im}(f) \cap (\overline{U}_0 \setminus W_0) \neq \emptyset$. Since the limit map f also has no degree zero connected components, (i) implies f can be split as $f = (f_1, f_2)$ where f_1 and f_2 map into \mathbb{P}_h and \mathbb{F}_0 respectively such that

$$\text{Im}(f_1) \cap V_0 = \text{Im}(f_2) \cap V_0.$$

It follows from (4.7) that $\text{Im}(f_1) \subset D$ and hence $\text{Im}(f_2) \cap V_0 = D \cap V_0$. Then, since the restriction of J_V on \mathbb{F}_0 is the product complex structure, $\text{Im}(f_2)$ lies in the section of \mathbb{F}_0 passing through the intersection point $D \cap V_0$. We have $\text{Im}(f) \subset W_0$ which contradicts (ii). \square

Fix $(J, \nu) \in \mathcal{J}(Z)$ sufficiently close to $(J_V, 0)$ and for small $|\lambda|$ set

$$\mathcal{M}_s^{\delta,*}(U_\lambda) = \{(f, s) \in \mathcal{M}_s^\delta(Z_\lambda) \mid f \in \mathcal{M}_{\chi, n, (1^d), (1^d)}^{\tilde{V}_\lambda, *}(U_\lambda, dS)\}.$$

Consider the restriction of the evaluation map (4.3):

$$ev_{s, U_0}^* : \bigcup \left(\mathcal{M}_{\chi_1, n_1, (1^d), s}^{V_1, V_0, *}(\mathbb{P}_h \cap U_0, dS) \times \mathcal{M}_{\chi_2, n_2, s, (1^d)}^{V_0, V_1, *}(\mathbb{F}_0 \cap U_0, dS) \right) \longrightarrow V_0^{\ell(s)} \times V_0^{\ell(s)}$$

where the union is over all $n_1 + n_2 = n$ and $\chi = \chi_1 + \chi_2 - 2\ell(s)$.

Remark 4.3. Let q be the intersection point of D and V in $Z_0 = \mathbb{P}_h \cup_{V_0} \mathbb{F}_0$. Then there is a unique section E_q of \mathbb{F}_0 that lies in $\mathbb{F}_0 \cap U_0$ and intersects with V_0 at the point q . Choose $\ell(s)$ points $\{q_j\}$ in $V_0 \cap (\mathbb{F}_0 \cap U_0)$ that are sufficiently close to q . Denote by

$$\mathcal{M}_{\chi_2, n_2, s, (1^d)}^{V_0, V_2, *}(\mathbb{F}_0, dS) \cap \{q_j\}$$

the cut-down moduli space of $(V_0 \sqcup V_2)$ -regular (J, ν) -holomorphic maps f with $\ell(s)$ contact points $\{x_{n+j}\}$ (with V_0) satisfying $f(x_{n+j}) = q_j$. Since every holomorphic map representing the class dS and passing through the point q has its image in E_q , by the Gromov Compactness Theorem, we have

$$\mathcal{M}_{\chi_2, n_2, s, (1^d)}^{V_0, V_2, *}(\mathbb{F}_0 \cap U_0, dS) \cap \{q_j\} = \mathcal{M}_{\chi_2, n_2, s, (1^d)}^{V_0, V_2, *}(\mathbb{F}_0, dS) \cap \{q_j\}. \quad (4.8)$$

This shows that local invariants of \mathbb{F}_0 counting maps into U_0 with point constraints equal to the standard invariants of \mathbb{F}_0 with points constraints.

Let $\mathcal{K}_{s, U_0}^{\delta,*} = \mathcal{K}_s^\delta \cap (ev_{s, U_0}^*)^{-1}(\triangle_s)$. Lemma 4.2 and (4.4) imply

$$\begin{aligned} \lim_{\lambda \rightarrow 0} \bigsqcup_s \mathcal{M}_s^{\delta,*}(U_\lambda) &\subset \bigsqcup_s \mathcal{K}_{s, U_0}^{\delta,*} \\ \lim_{\lambda \rightarrow 0} \bigsqcup_s \left(\mathcal{M}_s^\delta(Z_\lambda) \setminus \mathcal{M}_s^{\delta,*}(U_\lambda) \right) &\cap \bigsqcup_s \mathcal{K}_{s, U_0}^{\delta,*} = \emptyset. \end{aligned} \quad (4.9)$$

Consequently, by (4.5), (4.6) and (4.9), for the restriction Φ_λ^{loc} of the smoothing map Φ_λ to

$$\mathcal{K}_{s, U_0}^{\delta,*}(U_\lambda) = \pi_{s, \lambda}^{-1}(\mathcal{K}_{s, U_0}^{\delta,*})$$

we have a commutative diagram (up to homotopy):

$$\begin{array}{ccc}
\sqcup_s \mathcal{K}_{s,U_0}^{\delta,*}(U_\lambda) & \xrightarrow{\Phi_\lambda^{loc}} & \sqcup_s \mathcal{M}_s^{\delta,*}(U_\lambda) \\
\downarrow st_\lambda & & \downarrow st \\
\sqcup_s \widetilde{\mathcal{M}}_{\chi_1, n_1 + \ell(s) + d} \times \widetilde{\mathcal{M}}_{\chi_2, n_2 + \ell(s) + d} & \xrightarrow{\xi} & \widetilde{\mathcal{M}}_{\chi, n + 2d}
\end{array} \quad (4.10)$$

Step 3 : The commutative diagram (4.10) leads to a sum formula for local invariants for the sum $(\mathbb{P}_h, V_1 \sqcup V_2)$ of $(\mathbb{P}_h, V_1 \sqcup V_0)$ and $(\mathbb{F}_0, V_0 \sqcup V_2)$ along V_0 . We first assume that all maps in

$$\mathcal{M}_{\chi, n, (1^d), (1^d)}^{\tilde{V}_\lambda, *} (U_\lambda, dS) \quad (4.11)$$

are δ -flat when $|\lambda|$ is small. For fixed $n_1 + n_2 = n$, one can choose a continuous family of geometric representatives B_λ disjoint with \tilde{V}_λ satisfying:

- each B_λ ($\lambda \neq 0$) is a geometric representative of the n product of fiber classes $F^{\otimes n} = F \otimes \cdots \otimes F$ of $(Z_\lambda)^n \simeq (\mathbb{P}_h)^n$,
- $B_0 = B_{\mathbb{P}_h} \sqcup B_{\mathbb{F}_0}$ where $B_{\mathbb{P}_h}$ and $B_{\mathbb{F}_0}$ are geometric representatives of the classes $F^{\otimes n_1}$ of $(\mathbb{P}_h)^{n_1}$ and $F^{\otimes n_2}$ of $(\mathbb{F}_0)^{n_2}$ respectively.

It now follows from the diagram (4.10) that

$$\begin{aligned}
& [st(\mathcal{M}_{\chi, n, (1^d), (1^d)}^{\tilde{V}_\lambda, *} (U_\lambda, dS) \cap B_\lambda)] \\
&= \sum_s \frac{|s|}{\ell(s)!} (\xi_{\ell(s)})_* [st((ev_{s,U_0}^*)^{-1}(\Delta_s) \cap B_0)] \in H_*(\widetilde{\mathcal{M}}_{\chi, n+2d})
\end{aligned} \quad (4.12)$$

where the sum is over all $\ell(s) = d$. Here, the factor $|s|$ is the degree of the covering map $\pi_{s,\lambda}$, the factor $\ell(s)!$ reflects the fact that each map in the space $\mathcal{M}_s^{\delta,*}(U_\lambda)$ in the diagram (4.10) is a labeled map as in (4.2) and the classes $[st(\cdot)]$ are defined by the images of cut-down moduli spaces under stabilization map as in Remark 3.4.

In general, if there are maps in the space (4.11) that are not δ -flat then there is a correction term in (4.12) given by three S -matrices in \mathbb{F}_0 (cf. Definition 11.3 of [IP2]) for V_1, V_0 and V_2 . In our case, since the constraint $(F^*)^n$ is supported off the neck, Lemma 14.6 of [IP2] and Theorem 12.3 of [IP2] imply that the correction term is trivial. Consequently, by the splitting of the diagonal Δ_s , together with (3.4), (4.8) and Remark 3.1, it follows from (4.12) that

$$\begin{aligned}
& GT_{(1^d), (1^d)}^{loc, h, p} \left(\prod_{i=1}^n \phi_i^{k_i}(F^*) \right) = GT_{(1^d), (1^d), \chi}^{loc, h, p} ((F^*)^n) \cap \prod_{i=1}^n \phi_i^{k_i}(F^*) \\
&= \sum \frac{|m|}{m!} GT_{(1^d), m, \chi_1}^{loc, h, p} ((F^*)^{n_1}) \otimes GT_{m, (1^d), \chi_2}^{\mathbb{F}_0} ((F^*)^{n_2}) \cap (\xi_{\ell(m)})^* \left(\prod_{i=1}^n \phi_i^{k_i} \right)
\end{aligned} \quad (4.13)$$

where the sum is over all partitions m of d and $\chi_1 + \chi_2 - 2\ell(m) = \chi$ (cf. Theorem 12.3 of [IP2]).

Proof of Theorem A : Let $f = (f_1, f_2)$ be a map that contributes to the right hand-side of (4.13). Then we have

- every connected component of the domain of $(V_0 \sqcup V_i)$ -regular map f_i has at least two contact (marked) points with $V_0 \sqcup V_i$,
- every connected component of the domain of f_2 has exactly one contact marked point with V_0 ; the contact constraint with V_0 is $C_{pt^{\ell(m)}}$ (see (3.7)) and the image of each connected component can't pass through more than two distinct points on V_0 .

Noting the gluing map $\xi_{\ell(m)}$ is the map obtained by successively applying gluing maps as in (1.3) to connected components, by Lemma 1.2 (b) we have

$$(\xi_{\ell(m)})^* \left(\prod_{i=1}^n \phi_i^{k_i} \right) = \prod_{i=1}^{n_1} \phi_i^{k_i} \otimes \prod_{i=1}^{n_2} \phi_i^{k_{n_1+i}}. \quad (4.14)$$

The sum formula (4.13) together with (4.14) gives

$$GT_{(1^d), (1^d)}^{loc, h, p} \left(\prod_{i=1}^n \phi_i^{k_i} (F^*) \right) = \sum_m \frac{|m|}{m!} GT_{(1^d), m}^{loc, h, p} \left(\prod_{i=1}^{n_1} \phi_i^{k_i} (F^*) \right) \cdot GT_{m, (1^d)}^{\mathbb{F}_0} \left(\prod_{i=1}^{n_2} \phi_i^{k_{n_1+i}} (F^*) \right). \quad (4.15)$$

Now, Theorem A follows from (4.15) and Proposition 5.1 in the next section. \square

5 Descendent Invariants vs. Relative Invariants

The aim of this section is to show :

Proposition 5.1. $GT_d^{loc, h, p} \left(\prod_{i=1}^n \tau_{k_i} (F^*) \right) = \frac{1}{(d!)^2} GT_{(1^d), (1^d)}^{loc, h, p} \left(\prod_{i=1}^n \phi_i^{k_i} (F^*) \right).$

By the relation of GT and GW invariants (cf. Section 2 of [IP2]), it suffices to prove Proposition 5.1 for local GW invariants that count maps with connected domains. Let $GW_d^{loc, h, p}(\cdot)$ and $GW_{m^1, m^2}^{loc, h, p}(\cdot)$ denote absolute and relative local GW invariants and let

$$\pi = \pi_{2d} : \overline{\mathcal{M}}_{g, n+2d} \rightarrow \overline{\mathcal{M}}_{g, n} \quad (5.1)$$

be the forgetful map that forgets the last $2d$ marked points.

Lemma 5.2. *If $\sum k_i = 0$ or $n \geq 3$, then we have*

$$GW_d^{loc, h, p} \left(\prod_{i=1}^n \phi_i^{k_i} (F^*) \right) = \frac{1}{(d!)^2} GW_{(1^d), (1^d)}^{loc, h, p} \left(\prod_{i=1}^n \pi^* \phi_i^{k_i} (F^*) \right). \quad (5.2)$$

Proof. Consider the symplectic fiber sum $\mathbb{P}_h = \mathbb{F}_0 \#_{V_1} \mathbb{P}_h \#_{V_2} \mathbb{F}_0$ where V_1 and V_2 are two distinct fibers of \mathbb{P}_h . This sum can be obtained by blowing up $\mathbb{P}_h \times \mathbb{C}$ along $(V_1 \cup V_2) \times \{0\}$. The same arguments of Section 4 thus give a sum formula that is analogous to (4.13) :

$$\begin{aligned} & GT_d^{loc, h, p} \left(\prod_{i=1}^n \phi_i^{k_i} (F^*) \right) \\ &= \sum \frac{|m^1| |m^2|}{m^1! m^2!} GT_{m^1, \chi_1}^{\mathbb{F}_0} \otimes GT_{m^1, m^2, \chi_0}^{loc, h, p} \left((F^*)^n \right) \otimes GT_{m^2, \chi_2}^{\mathbb{F}_0} \cap (\xi_{\ell(m^1), \ell(m^2)})^* \left(\prod_{i=1}^n \phi_i^{k_i} \right) \end{aligned} \quad (5.3)$$

where the sum is over all $\chi_1 + \chi_0 + \chi_2 - 2\ell(m^1) - 2\ell(m^2) = 2d(1-h) - 2\sum k_i$ and $\xi_{\ell(m^1), \ell(m^2)}$ is the gluing map obtained by identifying contact points of domains (see above Theorem 4.1). If $\sum k_i = 0$ then for $k = 1, 2$, by dimension count, we have

$$0 = 2d - \frac{1}{2}\chi_k + (\ell(m^k) - d) - \ell(m^k) = d - \frac{1}{2}\chi_k. \quad (5.4)$$

This shows $\chi_k = 2d$ and hence $m^k = (1^d)$. Thus, (5.2) for $n = 0$ follows from (3.8) and (5.3).

Assume $n \geq 3$ and let $f = (f_1, f_0, f_2)$ be a map that contributes to the right hand side of (5.3). In order to obtain a sum formula for local GW invariants, we assume that the domain of f is connected. We have

- since all marked points of f map into the middle \mathbb{P}_h side, the domain of f_k ($k = 1, 2$) mapped into \mathbb{F}_0 has no marked points except contact points,
- as in the proof of Theorem A, every connected components of the domain of f_k has one contact point with V_k .

It follows that the domain of f_0 is connected and the gluing map $\xi = \xi_{\ell(m^1), \ell(m^2)}$ can be obtained by composing gluing maps as in (1.3) with $n_2 = 0$:

$$\eta : \overline{\mathcal{M}}_{g_1, n_1+1} \times \overline{\mathcal{M}}_{g_2, 1} \rightarrow \overline{\mathcal{M}}_{g_1+g_2, n_1+1}$$

where $n_1 \geq n \geq 3$. By Lemma 1.2 (a) for $g_2 = 0$ and by Lemma 1.2 (b) for $g_2 \geq 1$, one can see that the pull-back class $\xi^*\phi_i$ restricts to the trivial class on two \mathbb{F}_0 sides and hence f_k is constrained by only $\ell(m^k)$ point contact constraints. The dimension count (5.4) then shows $\chi_k = 2d$ and $m^k = (1^d)$, and hence ξ is a composition of gluing maps η as above with $(g_2, n_2) = (0, 0)$. Consequently, again by Lemma 1.2 (a), we have

$$\xi^*\phi_i = 1 \otimes (\phi_i - \sum \delta_{\{i\} \cup I}) \otimes 1 \quad (5.5)$$

where the sum is over all $I \subset \{n+1, \dots, n+2d\}$ with $I \neq \emptyset$. On the other hand, (1.2) shows

$$\pi^*\phi_i = \phi_i - \sum \delta_{\{i\} \cup I}. \quad (5.6)$$

Now, (5.2) for $n \geq 3$ follows from (3.8), (5.3), (5.5) and (5.6). \square

Remark 5.3. The same argument of the proof of Lemma 5.2 applies to various sum formulas for the fiber sum of \mathbb{P}_h and \mathbb{F}_0 . In particular, for dimension zero local invariants (i.e. $\sum k_i = 0$), one can use (3.8) and the dimension count (5.4) to show

$$GT_d^{loc, h, p} = \frac{1}{d!} GT_{(1^d)}^{loc, h, p} = \frac{1}{(d!)^2} GT_{(1^d), (1^d)}^{loc, h, p} \quad \text{and} \quad GT_m^{loc, h, p} = \frac{1}{d!} GT_{m, (1^d)}^{loc, h, p}$$

Lemma 5.4. *If $\sum k_i = 0$ or $n \geq 3$, then we have*

$$GW_d^{loc, h, p} \left(\prod_{i=1}^n \tau_{k_i}(F^*) \right) = \frac{1}{(d!)^2} GW_{(1^d), (1^d)}^{loc, h, p} \left(\prod_{i=1}^n \phi_i^{k_i}(F^*) \right). \quad (5.7)$$

Proof. If $\sum k_i = 0$ then (5.7) follows from Lemma 5.2. Assume $n \geq 3$ and $h > 0$ (we will give a proof for the case when $h = 0$ in the appendix). Let $V = V_1 \cup V_2$ be a union of two distinct fibers of \mathbb{P}_h and denote by

$$\mathcal{M}^V = \mathcal{M}_{g,n,(1^d),(1^d)}^V(U, dS)$$

the local relative GW moduli space. Let B be a product of n generic fibers of \mathbb{P}_h each of which is disjoint with V and let f be a limit map of a sequence in the cut-down moduli space $C\mathcal{M}^V \cap B$ where $C\mathcal{M}^V$ is the closure of \mathcal{M}^V in $\overline{\mathcal{M}}_{g,n+2d}(U, dS)$. Then, Remark 2.5 shows that every genus zero irreducible component of f maps entirely into either B or V . This implies that for $1 \leq i \leq n$ and for any $I \subset \{n+1, \dots, n+2d\}$ with $I \neq \emptyset$ we have

$$[\mathcal{M}^V] \cap \delta_{\{i\} \cup I} \otimes (F^*)^n = 0. \quad (5.8)$$

Therefore, (5.7) for $h > 0$ follows from (2.7), Lemma 5.2, (5.6) and (5.8). \square

Proof of Proposition 5.1 : We will show that (5.7) holds for all n . By (4.15), we have

$$\begin{aligned} GW_{(1^d),(1^d)}^{loc,h,p} \left(\prod_{i=1}^n \phi_i^{k_i}(F^*) F^* F^* \right) &= \frac{1}{d!} GW_{(1^d),(1^d)}^{loc,h,p} \left(\prod_{i=1}^n \phi_i^{k_i}(F^*) \right) \cdot GT_{(1^d),(1^d)}^{\mathbb{F}_0} (F^* F^*) \\ &= d^2 GW_{(1^d),(1^d)}^{loc,h,p} \left(\prod_{i=1}^n \phi_i^{k_i}(F^*) \right) \end{aligned} \quad (5.9)$$

where the second equality follows from Divisor Axiom and (3.8). On the other hand, one can see that the generalized Divisor Axiom (cf. Lemma 1.4 of [KM]) for descendant GW invariants also holds for descendant local invariants. Thus, we have

$$\begin{aligned} GW_d^{loc,h,p} \left(\prod_{i=1}^n \tau_{k_i}(F^*) \right) &= \frac{1}{d^2} GW_d^{loc,h,p} \left(\prod_{i=1}^n \tau_{k_i}(F^*) F^* F^* \right) \\ &= \frac{1}{d^2 (d!)^2} GW_{(1^d),(1^d)}^{loc,h,p} \left(\prod_{i=1}^n \phi_i^{k_i}(F^*) F^* F^* \right) = \frac{1}{(d!)^2} GW_{(1^d),(1^d)}^{loc,h,p} \left(\prod_{i=1}^n \phi_i^{k_i}(F^*) \right). \end{aligned}$$

where the second equality follows from Lemma 5.4 and the last from (5.9). This completes the proof of Proposition 10.1. \square

6 Local Contributions to GT invariants of Ruled Surfaces

Let $\pi : \mathbb{F}_1 = \mathbb{P}(\mathcal{O}_E(1) \oplus \mathcal{O}_E) \rightarrow E$ be a ruled surface over $E = \mathbb{P}^1$. This section describes local contributions to GW invariants of \mathbb{F}_1 that is needed for the proof of Theorem B. We also denote by E the zero section of \mathbb{F}_1 and by S the section class represented by the zero section E .

Remark 6.1. Let $f : C \rightarrow \mathbb{F}_1$ be a holomorphic map from a smooth domain C that represents the class dS . Then f defines a holomorphic section ξ of the line bundle $(\pi \circ f)^* \mathcal{O}_E(1)$ over C . The zero set $Z(\xi)$ of ξ is the preimage $f^{-1}(E)$ of the zero section E , so if the image of f does not lie in E (i.e. $\xi \neq 0$) then

$$|f^{-1}(E)| = |Z(\xi)| \leq \#Z(\xi) = \deg((\pi \circ f)^* \mathcal{O}_E(1)) = d$$

where $\#Z(\xi)$ is the number of zeros of ξ counted with multiplicities.

Choose distinct fibers V_1 , V_0 and V_2 of \mathbb{F}_1 and set

$$V = V_1 \sqcup V_0 \sqcup V_2.$$

Remark 6.2. Let $f : \mathbb{P}^1 \rightarrow \mathbb{F}_1$ be a holomorphic map that represents the class $2S$. If f has a contact vector (2) with V_i at $p_i \in \mathbb{P}^1$ then for the composition map

$$\mathbb{P}^1 \xrightarrow{f} \mathbb{F}_1 \xrightarrow{\pi} E = \mathbb{P}^1$$

the point p_i is a ramification point of multiplicity two. Thus $f^{-1}(V)$ consists of at least four points. For otherwise, $\pi \circ f$ is a holomorphic map of degree 2 with three ramification points, which is impossible by the Riemann-Hurwitz formula.

Remark 6.1 and Remark 6.2 immediately give :

Lemma 6.3. *Let f be a holomorphic map into \mathbb{F}_1 from a smooth domain of genus g representing the class dS and satisfying*

$$\text{Im}(f) \cap (V_1 \sqcup V_2) = (V_1 \sqcup V_2) \cap E.$$

If either (i) $d = 1$ or (ii) $d = 2$, $g = 0$ and the contact vector of f with V_0 is (2) then the image of f lies in E .

Fix a neighborhood U of the zero section E whose closure is disjoint from the infinity section of \mathbb{F}_1 . For V -compatible (J, ν) and ordered sequences s^1 and s^2 with $\deg(s^1) = \deg(s^2) = 2$, choose $\ell(s^1)$ points $\{p_i\}$ in V_1 and $\ell(s^2)$ points $\{q_j\}$ in V_2 and let

$$\mathcal{M}_{0,s^1,(2),s^2}^{V_1,V_0,V_2}(U, 2S, J, \nu) \cap \{p_i, q_j\} \subset \mathcal{M}_{0,s^1,(2),s^2}^{V_1,V_0,V_2}(U, 2S, J, \nu) \quad (6.1)$$

denote the cut-down moduli space of V -regular maps f into U with (connected) domain of genus zero and with n contact points $\{x_i\}$ satisfying: $f(x_0) \in V_0$, $f(x_i) = p_i$ and $f(x_{\ell(s^1)+j}) = q_j$ where $n = \ell(s^1) + \ell(s^2) + 1$. This cut-down moduli space has (formal) dimension zero. Consider a sequence of maps $(f_k, C_k; \{x_i^k\})$ in the cut-down moduli spaces

$$\mathcal{M}_{0,s^1,(2),s^2}^{V_1,V_0,V_2}(U, 2S, J_k, \nu_k) \cap \{p_i^k, q_j^k\} \quad (6.2)$$

where the points $\{p_i^k\} \sqcup \{q_j^k\} \subset (V_1 \sqcup V_2)$ converge to points in $(V_1 \sqcup V_2) \cap E$ and V -compatible (J_k, ν_k) converges to the complex structure of \mathbb{F}_1 as $k \rightarrow \infty$. By the Gromov Compactness Theorem, after passing to subsequences, the sequence of maps $(f_k, C_k; \{x_i^k\})$ then converges to a holomorphic map

$$(f, C; \{x_i\}) \in \overline{\mathcal{M}}_{0,n}(\overline{U}, 2S). \quad (6.3)$$

Since $\text{Im}(f) \subset \overline{U}$, every component of C mapped entirely into V is a ghost component. The following lemma shows that the image of f lies in the zero section E . Let $s^0 = (2)$.

Lemma 6.4. *Let $(f, C; \{x_i\})$ be as above and let C_i denote an irreducible component C that contains a marked point x_{i_0} mapped into V_i . If $s^i = (2)$ for some $0 \leq i \leq 2$ then we have*

(a) *if f is V_i -regular then the restriction f to C_i represents the class $2S$,*

(b) if f is not V_i -regular then C_i is a ghost component with $x_j \notin C_i$ for $j \neq i_0$ and $C \setminus C_i$ has two connected components C_i^ℓ such that the restriction of f to C_i^ℓ represents the class S .

In particular, the image of f lies in the zero section E of \mathbb{F}_1 .

Proof. First note that, since $\text{Im}(f_k) \rightarrow \text{Im}(f)$ and $\text{Im}(f) \subset \overline{U}$, we have

(i) $\text{Im}(f) \cap (V_1 \sqcup V_2) = (V_1 \sqcup V_2) \cap E$,

(ii) the restriction of f to each component of C represents the class kS where $0 \leq k \leq 2$.

If a marked point x_i is a limit point of the contact points x_i^k of f_k with V_i , i.e. $f_k(x_i^k) \in V_i$, then x_i is a contact point of f with the same contact order as x_i^k unless a component containing x_i maps into V . Thus, if f is V_i -regular then f has a contact vector (2) with V_i at x_{i_0} and hence by (ii) the restriction of f to C_i represents the class $2S$. This proves (a).

On the other hand, if f is not V_i -regular then C_i is a ghost component mapped into V_i . Moreover, by the assumption $s^i = (2)$ only one marked point x_{i_0} maps into V_i , so $x_j \notin C_i$ for $j \neq i_0$. Note that, since C is a connected curve of (arithmetic) genus zero, each irreducible component of C is smooth of genus zero. Thus Remark 1.1 implies that $C \setminus C_i$ has at least two connected components. Let C_i^ℓ be a connected component of $C \setminus C_i$. Then,

$$|C_i^\ell \cap \overline{C \setminus C_i^\ell}| = |C_i^\ell \cap C_i| = 1$$

where the second equality follows from the fact C is a connected curve of genus zero. So, if C_i^ℓ maps to a point then $f(C_i^\ell) \in V_i$ and, by Remark 1.1, C_i^ℓ has at least two marked points x_j with $f(x_j) \in V_i$. This is impossible since $f(x_j) \in V_j$ for some V_j disjoint with V_i . Thus there are two connected components of $C \setminus C_i$ such that the restriction of f to both components represent the class S . This completes the proof (b).

Now (a), (b), (i), (ii) and Lemma 6.3 imply that the image of f lies in the zero section E since $s^0 = (2)$. \square

Remark 6.5. Let f be a limit map as in Lemma 6.4 and suppose that C_0 and C_1 are irreducible components of the domain C of f which contain marked points x_0 and x_1 mapped into V_0 and V_1 . Suppose $s^1 = (2)$. Then, since $s^0 = (2)$, Lemma 6.4 implies that f is V_0 -regular if and only if f is V_1 -regular. Suppose f is not $(V_0 \sqcup V_1)$ -regular. Then, by Lemma 6.4 (b), both C_0 and C_1 are ghost components with $C_0 \cap C_1 = \emptyset$. Since the domain C is a connected curve of genus zero, the same argument for the proof of Lemma 6.4 (b) shows that $C \setminus (C_0 \sqcup C_1)$ has at least three connected components C_{01}^ℓ such that the restriction of f to each C_{01}^ℓ represents the class S . This is impossible, so (i) f must be $(V_0 \sqcup V_1)$ -regular (ii) $C_0 = C_1$ and (iii) the restriction of f to the component $C_0 = C_1$ represents the class $2S$. In particular, if $s^1 = s^2 = (2)$ then $f^{-1}(V)$ consists of three contact points. This contradicts Remark 6.2. Therefore, if $s^1 = s^2 = (2)$ then for all large k the cut-down moduli space (6.2) must be empty.

For V -compatible (J, ν) and for points $\{p_i\}$ in V_1 and $\{q_j\}$ in V_2 denote by

$$\mathcal{M} = \mathcal{M}_{0, s^1, (2), s^2}^{V_1, V_0, V_2}(\mathbb{F}_1, 2S) \cap \{p_i, q_j\}$$

the cut-down moduli space of V -regular (J, ν) -holomorphic maps into \mathbb{F}_1 with the point constraints $\{p_i, q_j\}$. Then we have a splitting of contributions to relative GW invariant

$$GW_{\mathbb{F}_1, 2S, 0, s^1, (2), s^2}^{V_1, V_0, V_2}(C_{pt^\ell(s^1)}, C_F, C_{pt^\ell(s^2)}) = [\mathcal{M}] = [\mathcal{M}(U)] + [\mathcal{M} \setminus \mathcal{M}(U)] \quad (6.4)$$

where $\mathcal{M}(U)$ is the cut-down moduli space (6.1) and $\mathcal{M} \setminus \mathcal{M}(U)$ consists of maps f in \mathcal{M} whose image does not lie in $U \subset \mathbb{F}_1$. Here, $[\cdot]$ denotes the zero-dimensional homology class defined by cut-down moduli space as in Remark 3.4. In general, this splitting is not well-defined, namely it depends on the choice of (J, ν) and the point constraints $\{p_i, q_j\}$. However, by the Gromov Compactness theorem and Lemma 6.4, the splitting (6.4) is well-defined whenever (i) (J, ν) is close to the complex structure of \mathbb{F}_1 and (ii) points $\{p_i, q_j\}$ are also close to points in $(V_1 \sqcup V_2) \cap E$. In such cases, the contribution

$$[\mathcal{M}(U)] = [\mathcal{M}_{0,s^1,(2),s^2}^{V_1,V_0,V_2}(U, 2S; C_{pt^{\ell(s^1)}}, C_F, C_{pt^{\ell(s^2)}})] \quad (6.5)$$

is independent of the choice of (J, ν) and $\{p_i, q_j\}$. Furthermore, this local contribution is independent of the choice of the neighborhood U of the zero section E in \mathbb{F}_1 .

Lemma 6.6. *For any neighborhood U of the zero section E in \mathbb{F}_1 whose closure is disjoint from the infinity section of \mathbb{F}_1 , we have*

$$(a) \quad [\mathcal{M}_{0,(2),(2),(2)}^{V_1,V_0,V_2}(U, 2S; C_{pt}, C_F, C_{pt})] = 0 \quad (b) \quad [\mathcal{M}_{0,(2),(2),(1,1)}^{V_1,V_0,V_2}(U, 2S; C_{pt}, C_F, C_{pt^2})] = 1$$

Proof. (a) follows from Remark 6.5. Let $s^1 = (2)$, $s^2 = (1, 1)$ and $(f, C; \{x_i\})$ be a limit map as in (6.3). Again by Remark 6.5, f is $(V_1 \sqcup V_0)$ -regular map into $E \subset \mathbb{F}_1$ and the restriction of f to the component $C_0 = C_1$ containing x_0 and x_1 represents the class $2S$. Stability of f then implies either $C = C_0$ or $C = C_0 \cup C_2$ where C_2 is a ghost component containing the marked points x_2 and x_3 . Suppose $C = C_0 \cup C_2$. In this case, C has one node since the (arithmetic) genus of C is zero. The restriction $f_0 = f|_{C_0}$ has a contact order two with V_2 at the node of C , so $f_0^{-1}(V)$ consists of three contact points. This contradicts Remark 6.2. Therefore, f is a holomorphic map from $C = \mathbb{P}^1$ into $E = \mathbb{P}^1$ of degree two with two ramification points x_0 and x_1 and $f(x_i) \in V_2 \cap E$ for $i = 2, 3$. Observe that there is a unique such map f .

Let J denote the complex structure on \mathbb{F}_1 and set

$$\mathcal{M}^V = \mathcal{M}_{0,(2),(2),(1,1)}^{V_1,V_0,V_2}(U, 2S, J) \quad \text{and} \quad h = h_{(2)} \times h_{(1,1)} : \mathcal{M}^V \rightarrow V_1 \times (V_2 \times V_2)$$

where $h_{(2)}$ and $h_{(1,1)}$ are evaluation maps as in (3.2). Let D_f be the (full) linearization of holomorphic map equation at f . Since the normal bundle of $\text{Im}(f) = E$ is $\mathcal{O}_E(1)$, we have

- $\text{coker } D_f = H^1(f^*\mathcal{O}_E(1)) = H^1(\mathcal{O}_{\mathbb{P}^1}(2)) = 0$ and hence
- \mathcal{M}^V is smooth near f with $T_f\mathcal{M}^V = H^0(f^*\mathcal{O}_E(1)) = H^0(\mathcal{O}_{\mathbb{P}^1}(2))$,
- $dh_f(\xi) = (\xi(x_1), \xi(x_2), \xi(x_3))$.

In fact, regarding the neighborhood U of E in \mathbb{F}_1 as a disk subbundle of $\mathcal{O}_E(1)$, one can identify holomorphic sections ξ of $f^*U \subset f^*\mathcal{O}_E(1)$ with V -regular holomorphic maps f_ξ in \mathcal{M}^V — in local trivialization, $f_\xi(x) = (f(x), \xi(x))$. By Remark 6.1 the differential dh_f is one-to-one. Thus dh_f is onto since $h^0(\mathcal{O}_{\mathbb{P}^1}(2)) = 3$. We can now conclude that the contribution of $(f, \mathbb{P}^1; \{x_i\})$ to the invariant (6.5) is +1 since f has no nontrivial automorphisms and dh_f is onto and complex linear. This completes the proof of (b). \square

We will compute the local contribution (6.5) for the case $s^1 = s^2 = (1, 1)$ in the next section (see (7.10)).

7 Spin Curve Degeneration and Sum Formula I

This section proves Theorem B(a) in three steps. First we review the *dualizing sheaf*. The dualizing sheaf ω_X of a variety X (if exists) is the unique invertible sheaf making Serre duality valid; when X is smooth ω_X is the canonical bundle K_X . For a proper holomorphic map $f : X \rightarrow B$ between two smooth varieties, the relative dualizing sheaf ω_f is the locally free rank one sheaf $\omega_X \otimes (f^*\omega_B)^{-1}$ whose restriction to each fiber X_b is the dualizing sheaf of X_b (cf. [T] and [HM]).

Step 1 : Let $D_1 \cup_q D_2$ be a union of two smooth curves D_1 and D_2 of genera h_1 and h_2 , meeting at one point q . Blowing up the point q yields a nodal curve D_0 with an exceptional component $E = \mathbb{P}^1$ that meets $\bar{D} = D_1 \sqcup D_2$ at two points. A theta characteristic of the nodal curve D_0 is a line bundle N_0 together with a homomorphism $\phi : N_0^2 \rightarrow \omega_{D_0}$ satisfying :

- N_0 restricts to $\mathcal{O}(1)$ on the exceptional component E ,
- ϕ vanishes identically on E and restricts to an isomorphism $N_0^2|_{\bar{D}} \simeq \omega_{\bar{D}}$,

where ω_{D_0} and $\omega_{\bar{D}}$ are the dualizing sheaves of D_0 and \bar{D} respectively. Since $\omega_{\bar{D}}$ is the canonical bundle of \bar{D} , the line bundle N_0 restricts to theta characteristic N_1 on D_1 and restricts to theta characteristic N_2 on D_2 . The triple (D_0, N_0, ϕ) is a *spin curve* of genus $h = h_1 + h_2$ with parity $p \equiv p_1 + p_2 \pmod{2}$ where p_i is a parity of the spin curve (D_i, N_i) . It then follows from a universal deformation of the spin curve (D_0, N_0, ϕ) (cf. pg 570 [C]) that there are

- a family of curves $\rho : \mathcal{D} \rightarrow \Delta$ where Δ is a unit disk in \mathbb{C} , the fiber D_λ over $\lambda \neq 0$ is a smooth curve of genus h and the central fiber is the nodal curve D_0 ,
- a line bundle $\pi : \mathcal{N} \rightarrow \mathcal{D}$ together with a homomorphism $\Phi : \mathcal{N}^2 \rightarrow \omega_\rho$ such that each $(\mathcal{D}|_{\rho^{-1}(\lambda)}, \mathcal{N}|_{\rho^{-1}(\lambda)}, \Phi|_{\rho^{-1}(\lambda)})$ is a spin curve of genus h with parity p ,

where ω_ρ is the relative dualizing sheaf of ρ (for more details see [C]).

Let \mathcal{N}_{D_λ} ($\lambda \neq 0$) be the total space of $\mathcal{N}|_{D_\lambda}$ and $\mathcal{N}_{\bar{D}}$ be the total space of $\mathcal{N}|_{\bar{D}}$. Since both total spaces \mathcal{N}_{D_λ} and $\mathcal{N}_{\bar{D}}$ are smooth, there are short exact sequences

$$0 \rightarrow \pi^*\mathcal{N}|_{D_\lambda} \rightarrow T\mathcal{N}_{D_\lambda} \rightarrow \pi^*TD_\lambda \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \pi^*\mathcal{N}|_{\bar{D}} \rightarrow T\mathcal{N}_{\bar{D}} \rightarrow \pi^*T\bar{D} \rightarrow 0.$$

It then follows from these exact sequences that

$$K_{\mathcal{N}_{D_\lambda}} = \pi^*\mathcal{N}_{D_\lambda}^* \otimes \pi^*K_{D_\lambda} \quad \text{and} \quad K_{\mathcal{N}_{\bar{D}}} = \pi^*\mathcal{N}_{\bar{D}}^* \otimes \pi^*K_{\bar{D}}. \quad (7.1)$$

On the other hand, the homomorphism $\Phi : \mathcal{N}^2 \rightarrow \omega_\rho$ induces a homomorphism

$$\Phi' : \pi^*\mathcal{N} = \pi^*(\mathcal{N}^* \otimes \mathcal{N}^2) \longrightarrow \pi^*(\mathcal{N}^* \otimes \omega_\rho).$$

Let σ be the tautological section of $\pi^*\mathcal{N}$. Since the relative dualizing sheaf ω_ρ restricts to dualizing sheaf ω_{D_λ} on D_λ , by (7.1) the composition $\Phi' \circ \sigma$ is a section of $\pi^*(\mathcal{N}^* \otimes \omega_\rho)$ satisfying :

- $\Phi' \circ \sigma$ restricts to a holomorphic 2-form α_λ on \mathcal{N}_{D_λ} ($\lambda \neq 0$) whose zero set is D_λ ,

- $\Phi' \circ \sigma$ restricts to a holomorphic 2-form α_0 on $\mathcal{N}_{\bar{D}}$ whose zero set is \bar{D} .

Let \mathcal{N}_E denote the total space of $\mathcal{N}|_E$. For sufficiently small $\epsilon > 0$, choose a bump function β that is 1 on the complement of the 2ϵ -neighborhood of \mathcal{N}_E in $\mathcal{N}_{\mathcal{D}}$ and vanishes on the ϵ -neighborhood \mathcal{U}_ϵ of \mathcal{N}_E . For each point $x \in \mathcal{N}_{\mathcal{D}} \setminus \mathcal{U}_\epsilon$ we set $\beta\alpha_\lambda(v_x, \cdot) = 0$ for normal vectors v_x to submanifold \mathcal{N}_{D_λ} (or $\mathcal{N}_{\bar{D}}$ for $\lambda = 0$) at x and define $\alpha = \beta\alpha_\lambda$ at x . Extension α by zero then gives a 2-form on $\mathcal{N}_{\mathcal{D}}$ that restricts to $\beta\alpha_\lambda$ on \mathcal{N}_{D_λ} ($\lambda \neq 0$) and to $\beta\alpha_0$ on $\mathcal{N}_{\bar{D}}$.

Step 2 : Consider the projectivization $\mathbb{P}(\mathcal{N} \oplus \mathcal{O}_{\mathcal{D}})$ over \mathcal{D} that gives a degeneration

$$\lambda : Z = \mathbb{P}(\mathcal{N} \oplus \mathcal{O}_{\mathcal{D}}) \rightarrow \mathcal{D} \rightarrow \Delta$$

whose fiber Z_λ ($\lambda \neq 0$) is a ruled surface over D_λ isomorphic to $\mathbb{P}_h = \mathbb{P}(N \oplus \mathcal{O}_D)$ where (D, N) is a (smooth) spin curve of genus $h = h_1 + h_2$ with parity $p \equiv p_1 + p_2 \pmod{2}$ and whose central fiber Z_0 is the singular (ruled) surface

$$\mathbb{P}_{h_1} \cup_{V_1} \mathbb{F}_1 \cup_{V_2} \mathbb{P}_{h_2} \rightarrow D_0$$

where V_1 and V_2 are fibers over the nodes of D_0 . Note that the general fiber Z_λ ($\lambda \neq 0$) is the symplectic fiber sum

$$\mathbb{P}_h = \mathbb{P}_{h_1} \#_{V_1} \mathbb{F}_1 \#_{V_2} \mathbb{P}_{h_2}.$$

Let U be an (open) neighborhood of the zero section of $Z = \mathbb{P}(\mathcal{N} \oplus \mathcal{O}_{\mathcal{D}})$ and fix an isomorphism Ψ from U to some neighborhood of $\mathcal{D} \subset \mathcal{N}_{\mathcal{D}}$ taking the zero section of Z to \mathcal{D} . Choose a point q_0 in the exceptional component $E \subset D_0$ that is not a nodal point and let $B \subset \mathcal{D}$ be a normal disk to $E \subset \mathcal{D}$ at q_0 , namely the intersection $B \cap D_\lambda$ is one point for all small $|\lambda|$. Let \tilde{V}_λ be the fiber of $Z_\lambda \rightarrow D_\lambda$ over the intersection point $B \cap D_\lambda$ and set

$$V = V_1 \sqcup V_0 \sqcup V_2$$

where $V_0 = \tilde{V}_0$. Denote by $\mathcal{J}(Z)$ the space of all (J, ν) on Z satisfying: (i) each Z_λ is J -invariant and (ii) the restriction of (J, ν) to Z_0 and Z_λ ($\lambda \neq 0$) are V -compatible and \tilde{V}_λ -compatible respectively. We will use the same notation (J, ν) for its restriction to each Z_λ . Denote by the same S the section classes of \mathbb{P}_{h_1} , \mathbb{F}_1 , \mathbb{P}_{h_2} and Z_λ represented by the zero sections. Let D_i ($i = 1, 2$) and D_λ denote the zero sections of \mathbb{P}_{h_i} and Z_λ respectively. For each small $|\lambda|$ we set

$$U_\lambda = U \cap Z_\lambda.$$

By using the 2-form α on $\mathcal{N}_{\mathcal{D}}$ together with the isomorphism Ψ , we obtain:

Lemma 7.1. *There is an almost complex structure J_V on Z satisfying:*

- (a) $(J_V, 0) \in \mathcal{J}(Z)$ and J_V restricts to the complex structure of \mathbb{F}_1 ,
- (b) $\overline{\mathcal{M}}_{\chi, n}^*(U_\lambda, dS, J_V) = \overline{\mathcal{M}}_{\chi, n}^*(D_\lambda, d)$ and $\overline{\mathcal{M}}_{\chi, n}^*(\bar{U}_0 \cap \mathbb{P}_{h_i}, dS, J_V) = \overline{\mathcal{M}}_{\chi, n}^*(D_i, d)$,
- (c) for generic $(J, \nu) \in \mathcal{J}(Z)$ sufficiently close to $(J_V, 0)$ and for small $|\lambda| > 0$

$$[\mathcal{M}_{\chi, (2)}^{\tilde{V}_\lambda, *}(U_\lambda, 2S)] = GT_{(2)}^{loc, h, p} \quad \text{and} \quad [\mathcal{M}_{\chi_i, s_i}^{V_i, *}(\mathbb{P}_{h_i} \cap U_0, 2S)] = GT_{s_i}^{loc, h_i, p_i}$$

where $\chi = 2 - 4h$, $\chi_i = 2\ell(s_i) - 4h_i$ for $i = 1, 2$.

Proof. By the isomorphism Ψ as above, one can regard α_λ ($\lambda \neq 0$) and α_0 as holomorphic 2-forms on U_λ and on $U_0 \cap (\mathbb{P}_{h_1} \sqcup \mathbb{P}_{h_2})$ whose zero sets are D_λ and $D_1 \sqcup D_2$, respectively. Similarly, one can also regard β as a bump function on U and α as a 2-form on U satisfying (i) α vanishes on some neighborhoods of \mathbb{F}_1 , V and \tilde{V}_λ for small $|\lambda|$, and (ii) the restriction of α to U_λ and $U_0 \cap (\mathbb{P}_{h_1} \sqcup \mathbb{P}_{h_2})$ are respectively $\beta\alpha_\lambda$ and $\beta\alpha_0$. Now, let J_V be the almost complex structure on Z induced by α and the formula (0.1). Then, (a) follows from (i), (b) follows from (ii) and Remark 2.2 and (c) follows from definition and compactness by (b). \square

Remark 7.2. Let f be a map in $\mathcal{M}_{\chi,(2)}^{\tilde{V}_\lambda}(U_\lambda, 2S)$ with $f^{-1}(\tilde{V}_\lambda) = \{x\}$ where the Euler characteristic $\chi = 2 - 4h$. If there is a connected component of the domain of f that does not contain the contact point x then the restriction of f to that component represents the trivial homology class. The stability of f thus shows C is connected since there is no marked points except the contact point x . Consequently, for $\lambda \neq 0$ we have

$$\mathcal{M}_{\chi,(2)}^{\tilde{V}_\lambda,*}(U_\lambda, 2S) = \mathcal{M}_{\chi,(2)}^{\tilde{V}_\lambda}(U_\lambda, 2S) = \mathcal{M}_{g,(2)}^{\tilde{V}_\lambda}(U_\lambda, 2S) \quad (7.2)$$

where the genus $g = 2h$. Similar arguments also show that

$$\mathcal{M}_{\chi_0,s_1,(2),s_2}^{V_1,V_0,V_2,*}(\mathbb{F}_1, 2S) = \mathcal{M}_{g_0,s_1,(2),s_2}^{V_1,V_0,V_2}(\mathbb{F}_1, 2S) \quad (7.3)$$

where the Euler characteristic $\chi_0 = 2$ and the genus $g_0 = 0$.

Step 3 : Choose an (open) neighborhood W of the zero section of $Z = \mathbb{P}(\mathcal{N} \oplus \mathcal{O}_D)$ with $\overline{W} \subset U$ and for each small $|\lambda|$ set

$$W_\lambda = W \cap Z_\lambda.$$

The following is the key fact to the proof of Theorem B (a).

Lemma 7.3. *For $(J, \nu) \in \mathcal{J}(Z)$ sufficiently close to $(J_V, 0)$ and for small $|\lambda| > 0$, we have*

$$\mathcal{M}_{\chi,(2)}^{\tilde{V}_\lambda}(U_\lambda, 2S) \setminus \mathcal{M}_{\chi,(2)}^{\tilde{V}_\lambda}(W_\lambda, 2S) = \emptyset$$

where the Euler characteristic $\chi = 2 - 4h$.

Proof. Suppose not. Then, there exists a sequence of (J_k, ν_k) -holomorphic maps f_k into U_{λ_k} with $\text{Im}(f_k) \cap (U_{\lambda_k} \setminus W_{\lambda_k}) \neq \emptyset$ where $\lambda_k \rightarrow 0$ and (J_k, ν_k) converges to $(J_V, 0)$ as $k \rightarrow \infty$. After passing to subsequences, by the Gromov Compactness Theorem, f_k converges to a J_V -holomorphic map f into Z_0 such that (i) $\text{Im}(f) \subset \overline{U}_0$ and (ii) $\text{Im}(f) \cap (\overline{U}_0 \setminus W_0) \neq \emptyset$. By (i) and Lemma 7.1, f can be split as $f = (f_1, f_0, f_2)$ where f_1 and f_2 are respectively holomorphic maps into D_1 and D_2 , and f_0 is a holomorphic map into \mathbb{F}_1 such that

$$\text{Im}(f_0) \cap (V_1 \sqcup V_2) = (\text{Im}(f_1) \sqcup \text{Im}(f_2)) \cap (V_1 \sqcup V_2) = E \cap (V_1 \sqcup V_2). \quad (7.4)$$

Note that the domain C of f is a connected curve of genus $2h$ since by (7.2) f is a limit of maps with connected domains of genus $2h$. Also note that if f is not $(V_1 \sqcup V_2)$ -regular, there is a ghost component mapped into $V_1 \sqcup V_2$.

Let $f_{12} = (f_1, f_2)$, C_{12} be the domain of f_{12} and C_0 be the domain of f_0 . We can assume that C_0 contains all ghost components mapped into $V_1 \sqcup V_2$. Then f_{12} is $(V_1 \sqcup V_2)$ -regular, so $f_{12}^{-1}(V_1 \sqcup V_2) = (C_{12} \cap C_0)$. Let $\ell = |f_{12}^{-1}(V_1 \sqcup V_2)|$. Since $g - 1 = -\frac{1}{2}\chi$, we have

$$2h = g(C) = -\frac{1}{2}\chi(C_0) - \frac{1}{2}\chi(C_{12}) + \ell + 1. \quad (7.5)$$

Consider f_{12} as a holomorphic map into $D_1 \sqcup D_2$ and apply the Riemann-Hurwitz formula to each irreducible component of C_{12} . This gives

$$-\frac{1}{2}\chi(C_{12}) + \ell \geq 2h \quad (7.6)$$

since the geometric genus of each irreducible component is less than or equal to its arithmetic genus and f_{12} has at least $2(4 - \ell)$ ramification points. Consequently, by (7.5) and (7.6) we have

$$\chi(C_0) \geq 2. \quad (7.7)$$

Note that the image of f_0 does not lie in the zero section E of \mathbb{F}_1 by (ii) since f_{12} maps into $D_1 \cup D_2 \subset W_0$. Remark 6.1 and (7.4) imply that there is exactly one irreducible component C'_0 of C_0 such that the restriction $f'_0 = f|_{C'_0}$ has contact vectors (2) with V_1 and V_2 and all other irreducible components of C_0 are ghost components. Let C_0^1 be the connected component of C_0 that contains C'_0 . Since f has no degree zero components, we have

- all ghost components mapped into V_0 are contained in C_0^1 ,
- if there exists a connected component $C_0^2 \neq C_0^1$ of C_0 then C_0^2 is a union of ghost components such that C_0^2 has no marked points and maps into either V_1 or V_2 . Since $|C_0 \cap C_{12}| = \ell \leq 4$ and $C_0^1 \cap C_{12}$ contains at least one point mapped into V_1 and at least one point mapped into V_2 , we have $|C_0^2 \cap \overline{C \setminus C_0^1}| = |C_0^2 \cap C_{12}| = 1$ and hence $g(C_0^2) > 0$ by Remark 1.1.

Now from (7.7) we have $g(C_0^1) = 0$, so each irreducible component of C_0^1 has genus zero and no two irreducible components meet at more than one point. In particular, since C'_0 has genus zero, Lemma 6.3 implies that f'_0 has a contact vector (1, 1) with V_0 . In this case, since f is a limit map of a sequence of maps with contact order (2) with V_0 , there is a ghost component mapped into V_0 . Let C''_0 be a connected component of the union of all ghost components mapped into V_0 . Then, $C''_0 \subset C_0^1$, so $g(C''_0) = 0$ and

$$|C''_0 \cap \overline{C \setminus C''_0}| = |C''_0 \cap C'_0| < 2.$$

Since C''_0 has at most one marked point, we have a contradiction by Remark 1.1. \square

Proof of Theorem B (a) : The proof is identical to the proof of Theorem A. We only outline the proof. For (ordered) sequences s^i with $\deg(s_i) = 2$ where $i = 1, 2$, consider the evaluation map that records the intersection points with V_1 and V_2 :

$$\begin{aligned} ev_{s^1, s^2, U_0}^* : & \bigcup \mathcal{M}_{\chi_1, s^1}^{V_1, *}(\mathbb{P}_{h_1} \cap U_0, 2S) \times \mathcal{M}_{\chi_0, s^1, (2), s^2}^{V_1, V_0, V_2, *}(\mathbb{F}_1 \cap U_0, 2S) \times \mathcal{M}_{\chi_2, s^2}^{V_2, *}(\mathbb{P}_{h_2} \cap U_0, 2S) \\ & \longrightarrow \left(V_1^{\ell(s^1)} \times V_1^{\ell(s^1)} \right) \times \left(V_2^{\ell(s^2)} \times V_2^{\ell(s^2)} \right) \end{aligned}$$

where the union is over all $\chi_1 + \chi_0 + \chi_2 - 2\ell(s^1) - 2\ell(s^2) = 2 - 4h$. Let Δ_{s^i} be the diagonal of $V^{\ell(s^i)} \times V^{\ell(s^i)}$ for $i = 1, 2$. Lemma 7.3 and Theorem 10.1 of [IP2] then give

$$[\mathcal{M}_{\chi, (2)}^{\tilde{V}_\lambda, *}(U_\lambda, 2S)] = \sum_{s^1, s^2} \frac{|s^1||s^2|}{\ell(s^1)!\ell(s^2)!} [(ev_{s^1, s^2, U_0}^*)^{-1}(\Delta_{s^1} \times \Delta_{s^2})] \quad (7.8)$$

where $\chi = 2 - 4h$. On the other hand, the splitting of the diagonal $\Delta_{s^1} \times \Delta_{s^2}$ yields

$$\begin{aligned} & \sum_{s^1, s^2} \frac{|s^1||s^2|}{\ell(s^1)!\ell(s^2)!} [(ev_{s^1, s^2, U_0}^*)^{-1}(\Delta_{s^1} \times \Delta_{s^2})] \\ &= \sum_{m^1, m^2} \frac{|m^1||m^2|}{m^1!m^2!} GT_{m^1}^{loc, h_1, p_1} \cdot [\mathcal{M}_{0, m^1, (2), m^2}^{V_1, V_0, V_2}(\mathbb{F}_1 \cap U_0, 2S; C_{pt^{\ell(m^1)}}, C_F, C_{pt^{\ell(m^2)}})] \cdot GT_{m^2}^{loc, h_2, p_2} \\ &= GT_{(1^2)}^{loc, h_1, p_1} \cdot GT_{(2)}^{loc, h_2, p_2} + GT_{(2)}^{loc, h_1, p_1} \cdot GT_{(1^2)}^{loc, h_2, p_2} \\ &+ \frac{1}{4} GT_{(1^2)}^{loc, h_1, p_1} \cdot [\mathcal{M}_{0, (1^2), (2), (1^2)}^{V_1, V_0, V_2}(\mathbb{F}_1 \cap U_0, 2S; C_{pt^2}, C_F, C_{pt^2})] \cdot GT_{(1^2)}^{loc, h_2, p_2} \end{aligned} \quad (7.9)$$

where the first equality follows from (3.4), Lemma 7.1 (c), Remark 3.1 and (7.3) and the second equality follows from Lemma 6.6. Thus, by Lemma 7.1 (c), (7.8), (7.9), Remark 5.3 and Lemma 2.6, we have

$$\begin{aligned} GT_{(2)}^{loc, h, p} &= (-1)^{p_1} 2^{h_1} GT_{(2)}^{loc, h_2, p_2} + (-1)^{p_2} 2^{h_2} GT_{(2)}^{loc, h_1, p_1} \\ &+ (-1)^p 2^{h-2} [\mathcal{M}_{0, (1^2), (2), (1^2)}^{V_1, V_0, V_2}(\mathbb{F}_1 \cap U_0, 2S; C_{pt^2}, C_F, C_{pt^2})]. \end{aligned}$$

When $(h_2, p_2) = (0, +)$, this equation shows

$$[\mathcal{M}_{0, (1^2), (2), (1^2)}^{V_1, V_0, V_2}(\mathbb{F}_1 \cap U_0, 2S; C_{pt^2}, C_F, C_{pt^2})] = -4 GT_{(2)}^{loc, 0, +}. \quad (7.10)$$

This completes the proof. \square

8 Spin Curve Degeneration and Sum Formula II

This section proves Theorem B (b). Let $h \geq 2$ or $(h, p) = (1, +)$ and let $\mathcal{D} \rightarrow \Delta$ denote a family of curves over the unit disk Δ in \mathbb{C} whose fiber over $\lambda \neq 0$ is a smooth curve D_λ of genus h and whose central fiber D_0 is a union of two smooth components \bar{D} and E of genera $h-1$ and 0 , meeting at two points. Fix a theta characteristic \bar{N} on \bar{D} with parity p . One can then find a line bundle $\mathcal{N} \rightarrow \mathcal{D}$ that restricts to a theta characteristic on D_λ with parity p , to the theta characteristic \bar{N} on \bar{D} and to $\mathcal{O}(1)$ on E (cf. pg 570 [C]). The projectivization $\mathbb{P}(\mathcal{N} \oplus \mathcal{O}_{\mathcal{D}})$ gives a degeneration

$$\mathbb{P}(\mathcal{N} \oplus \mathcal{O}_{\mathcal{D}}) \rightarrow \mathcal{D} \rightarrow \Delta$$

such that (i) the general fiber Z_λ ($\lambda \neq 0$) is a ruled surface isomorphic to $\mathbb{P}_h = \mathbb{P}(N \oplus \mathcal{O}_D)$ where (D, N) is a smooth spin curve of genus h with parity p and (ii) the central fiber is the singular (ruled) surface

$$\mathbb{P}_{h-1} \bigcup_{V_1 \sqcup V_2} \mathbb{F}_1 \rightarrow D_0$$

where $\mathbb{P}_{h-1} = \mathbb{P}(\tilde{N} \oplus \mathcal{O}_{\tilde{D}})$ and V_1 and V_2 are fibers over the nodes of D_0 . Note that the general fiber Z_λ ($\lambda \neq 0$) is the symplectic fiber sum

$$\mathbb{P}_h = \mathbb{P}_{h-1} \#_{V_1 \sqcup V_2} \mathbb{F}_1.$$

Proof of Theorem B (b): The proof is also identical to those of Theorem A and Theorem B (a). We only sketch the proof. Fix a normal disk $B \subset \mathcal{D}$ to E at some point that is not a nodal point of D_0 and for each small $|\lambda|$, let \tilde{V}_λ be the fiber of $Z_\lambda \rightarrow D_\lambda$ over the intersection point of B and D_λ . Choose small neighborhoods U and W of the zero section of $Z = \mathbb{P}(\mathcal{N} \oplus \mathcal{O}_{\mathcal{D}})$ satisfying $\overline{W} \subset U$ and set

$$W_\lambda = W \cap Z_\lambda \quad \text{and} \quad U_\lambda = U \cap Z_\lambda.$$

The same arguments as in Lemma 7.1 and Lemma 7.3 then show that the tautological section of $\pi^* \mathcal{N}$ over the total space of $\pi : \mathcal{N} \rightarrow \mathcal{D}$ induces an almost complex structure J_ν on Z satisfying: for (J, ν) sufficiently close to $(J_\nu, 0)$ and for small $|\lambda| > 0$

$$GT_{(2)}^{loc, h, p} = [\mathcal{M}_{\chi, (2)}^{\tilde{V}_\lambda, *}(U_\lambda, 2S)] \quad \text{and} \quad GT_{s^1, s^2}^{loc, h-1, p} = [\mathcal{M}_{\chi_0, s^1, s^2}^{V_1, V_2, *}(\mathbb{P}_{h-1} \cap U_0)] \quad (8.1)$$

$$\mathcal{M}_{\chi, (2)}^{\tilde{V}_\lambda, *}(U_\lambda, 2S) \setminus \mathcal{M}_{\chi, (2)}^{\tilde{V}_\lambda, *}(W_\lambda, 2S) = \emptyset \quad (8.2)$$

where $\chi = 2 - 4h$ and $\chi_0 = -4h + 2 \sum \ell(s^i)$. For ordered sequences s^i with $\deg(s^i) = 2$ where $i = 1, 2$, consider the evaluation map that records the intersection points with V_1 and V_2 :

$$\begin{aligned} ev_{s^1, s^2, U_0}^* : \bigcup \mathcal{M}_{\chi_0, s^1, s^2}^{V_1, V_2, *}(\mathbb{P}_{h-1} \cap U_0, 2S) \times \mathcal{M}_{\chi_1, s^1, (2), s^2}^{V_1, V_0, V_2, *}(\mathbb{F}_1 \cap U_0, 2S) \\ \longrightarrow \left(V_1^{\ell(s^1)} \times V_1^{\ell(s^1)} \right) \times \left(V_2^{\ell(s^2)} \times V_2^{\ell(s^2)} \right) \end{aligned}$$

where $V_0 = \tilde{V}_0$, the union is over all $\chi_1 + \chi_0 - 2\ell(s^1) - 2\ell(s^2) = 2 - 4h$. Let \triangle_{s^i} be the diagonal of $V^{\ell(s^i)} \times V^{\ell(s^i)}$ where $i = 1, 2$. Then we have

$$\begin{aligned} GT_{(2)}^{loc, h, p} &= [\mathcal{M}_{\chi, (2)}^{\tilde{V}_\lambda, *}(U_\lambda, 2S)] = \sum_{s^1, s^2} \frac{|s^1| |s^2|}{\ell(s^1)! \ell(s^2)!} [(ev_{s^1, s^2, U_0}^*)^{-1}(\triangle_{s^1} \times \triangle_{s^2})] \\ &= \sum_{m^1, m^2} \frac{|m^1| |m^2|}{m^1! m^2!} GT_{m^1, m^2}^{loc, h-1, p} \cdot [\mathcal{M}_{0, m^1, (2), m^2}^{V_1, V_0, V_2}(\mathbb{F}_1 \cap U_0, 2S; C_{pt^{\ell(m^1)}}, C_F, C_{pt^{\ell(m^2)}})] \quad (8.3) \end{aligned}$$

where the first sum is over all ordered sequences s^1 and s^2 with $\deg(s^1) = \deg(s^2) = 2$ and the second sum is over all partitions m^1 and m^2 of 2; the first equality follows from (8.1), the second equality from (8.2) and Theorem 10.1 of [IP2] and the third equality from (3.4), (8.1), Remark 3.1 and (7.3). On the other hand, by Remark 5.3 we have

$$GT_{(1^2), (1^2)}^{loc, h-1, p} = 2^2 GT_2^{loc, h-1, p} \quad \text{and} \quad GT_{(1^2), (2)}^{loc, h-1, p} = 2 GT_{(2)}^{loc, h-1, p}.$$

This together with (8.3), Lemma 2.6, Lemma 6.6 and (7.10) completes the proof. \square

9 Reduction to Genus Zero Spin Curve Invariants

As described in the Introduction, Kiem and Li proved the Maulik-Pandharipande formulas (0.2) by reducing higher genus spin curve invariants to genus zero spin curve invariants. The aim of this section is to show how their reduction follows from Theorem A and Theorem B.

Proposition 9.1 ([KL1]).

$$(a) \quad GT_1^{loc,h,p} \left(\prod_{i=1}^n \tau_{k_i}(F^*) \right) = (-1)^p GT_1^{loc,0,+} \left(\prod_{i=1}^n \tau_{k_i}(F^*) \right)$$

$$(b) \quad GT_2^{loc,h,p} \left(\prod_{i=1}^n \tau_{k_i}(F^*) \right) = (-1)^p 2^h GT_2^{loc,0,+} \left(\prod_{i=1}^n \tau_{k_i}(F^*) \right)$$

Proof. The sum formula (0.3) for $d = 1, 2$ and Remark 5.3 show that

$$GT_1^{loc,h,p} \left(\prod_{i=1}^n \tau_{k_i}(F^*) \right) = GT_1^{loc,h,p} \cdot GT_{(1),(1)}^{\mathbb{F}_0} \left(\prod_{i=1}^n \phi_i^{k_i}(F^*) \right) \quad (9.1)$$

$$GT_2^{loc,h,p} \left(\prod_{i=1}^n \tau_{k_i}(F^*) \right) = \frac{1}{2} GT_2^{loc,h,p} \cdot GT_{(1,1),(1,1)}^{\mathbb{F}_0} \left(\prod_{i=1}^n \phi_i^{k_i}(F^*) \right) \\ + GT_{(2)}^{loc,h,p} \cdot GT_{(2),(1,1)}^{\mathbb{F}_0} \left(\prod_{i=1}^n \phi_i^{k_i}(F^*) \right). \quad (9.2)$$

Thus, Proposition 9.1 (a) follows from Lemma 2.6 and (9.1). Similarly, by Lemma 2.6 and (9.2), in order to prove Proposition 9.1 (b), we need to show

$$GT_{(2)}^{loc,h,p} = (-1)^p 2^h GT_{(2)}^{loc,0,+}. \quad (9.3)$$

The sum formula (0.4) for the case $(h_2, p_2) = (1, +)$ gives

$$GT_{(2)}^{loc,h,p} = (-1)^p 2^{h-1} GT_{(2)}^{loc,1,+} + 2 GT_{(2)}^{loc,h-1,p} - (-1)^p 2^h GT_{(2)}^{loc,0,+} \quad (9.4)$$

where $h \geq 2$. Applying the sum formula (0.4) twice with $p_1 = p_2 = \pm 1$ and $h_1 = h_2 = 1$ gives

$$GT_{(2)}^{loc,1,+} = -GT_{(2)}^{loc,1,-}.$$

This together with the sum formula (0.5) for the case $(h, p) = (1, +)$ yields

$$GT_{(2)}^{loc,1,p} = (-1)^p 2 GT_{(2)}^{loc,0,+}. \quad (9.5)$$

Using induction on genus h together with (9.4) and (9.5) then shows (9.3). This completes the proof. \square

Remark 9.2. The proof of Proposition 9.1 (b) by Kiem and Li (see Section 4 of [KL1]) goes as follows: they first obtained a sum formula similar to (9.2) using their sum formula and then showed (9.3) by calculating the local invariants $GT_2^{loc,h,p}(\tau(F^*))$ for all $h \geq 0$ using explicit algebro-geometric arguments.

10 Appendix

Let $GW_{d,g}(\cdot)$ and $GW_{(1^d),(1^d),g}(\cdot)$ respectively denote the absolute GW invariants of \mathbb{P}_0 for the class dS with genus g and the relative GW invariants of \mathbb{P}_0 relative to distinct fibers V_1 and V_2 of \mathbb{P}_h with contact constraint $C_{[V_i]^d}$ with V_i (we will omit the fibers V_i and the contact constraints $C_{[V_i]^d}$ in notation). Since local invariants of spin curve of genus $h = 0$ are GW invariants of \mathbb{P}_0 , the lemma below shows the formula (5.7) for the case when $h = 0$ and $n \geq 3$.

Lemma 10.1. *For $n \geq 3$, we have*

$$GW_{d,g}\left(\prod_{i=1}^n \tau_{k_i}(F^*)\right) = \frac{1}{(d!)^2} GW_{(1^d),(1^d),g}\left(\prod_{i=1}^n \phi_i^{k_i}(F^*)\right). \quad (10.1)$$

The proof consists of two steps.

Step 1 : We will relate the descendent classes for GW invariants of \mathbb{P}_0 to the ϕ_i classes. Following [KM], we set

$$\tau_{s_i} \phi_i^{t_i}(F^*) = \psi_i^{s_i} st^* \phi_i^{t_i} \cup ev_i^*(F^*).$$

Lemma 10.2. *Let $n \geq 3$. Then, for $s_j \geq 1$, we have*

$$\begin{aligned} GW_{d,g}\left(\prod_{i=1}^n \tau_{s_i} \phi_i^{t_i}(F^*)\right) &= GW_{d,g}\left(\prod_{i=1}^n \tau_{s_i - \delta_{ij}} \phi_i^{t_i + \delta_{ij}}(F^*)\right) \\ &- \sum_{0 < k < d} \delta_{ks_j} (-1)^{k-1} \frac{1}{k!} GW_{d-k,g}\left(\phi_j^{t_j}(F^*) \prod_{i \neq j} \tau_{s_i} \phi_i^{t_i}(F^*)\right). \end{aligned} \quad (10.2)$$

Proof. It follows from Theorem 1.1 of [KM] that for $s_j \geq 1$

$$\begin{aligned} GW_{d,g}\left(\prod_{i=1}^n \tau_{s_i} \phi_i^{t_i}(F^*)\right) &= GW_{d,g}\left(\prod_{i=1}^n \tau_{s_i - \delta_{ij}} \phi_i^{t_i + \delta_{ij}}(F^*)\right) \\ &+ \sum_{a, 0 < k < d} GW_{k,0}\left(\tau_{s_j-1}(F^*) H^a\right) GW_{d-k,g}\left(\phi_j^{t_j}(H_a) \prod_{i \neq j} \tau_{s_i} \phi_i^{t_i}(F^*)\right) \end{aligned} \quad (10.3)$$

where $\{H_a\}$ and $\{H^a\}$ are Poincaré dual basis of $H^*(\mathbb{P}_0)$. Fix a basis $\{1, S^* + F^*, F^*, \gamma^*\}$ and its dual basis $\{\gamma^*, F^*, S^*, 1\}$ of $H^*(\mathbb{P}_0)$ where γ^* be the Poincaré dual of the point class of \mathbb{P}_0 . Note that all degree zero ($d = 0$) invariants in the righthand side of (10.3) vanish since $n \geq 3$; no degree zero maps can pass through two distinct fibers. Moreover, for any $d > 0$ and g

$$GW_{d,g}(\gamma^* \cdots) = 0, \quad GW_{d,g}((S^* + F^*) \cdots) = 0, \quad GW_{d,g}(S^* \cdots) = -GW_{d,g}(F^* \cdots)$$

where the first follows from $S^2 = -1$, the second from $S(S + F) = 0$ and the third from the second. Consequently, (10.3) becomes

$$\begin{aligned} GW_{d,g}\left(\prod_{i=1}^n \tau_{s_i} \phi_i^{t_i}(F^*)\right) &= GW_{d,g}\left(\prod_{i=1}^n \tau_{s_i - \delta_{ij}} \phi_i^{t_i + \delta_{ij}}(F^*)\right) \\ &- \sum_{0 < k < d} GW_{k,0}\left(\tau_{s_j-1}(F^*) F^*\right) GW_{d-k,g}\left(\phi_j^{t_j}(F^*) \prod_{i \neq j} \tau_{s_i} \phi_i^{t_i}(F^*)\right). \end{aligned} \quad (10.4)$$

If $k \neq s_j$ then $GW_{k,0}(\tau_{s_j-1}(F^*)F^*) = 0$ by dimension count. So, it remains to show

$$GW_{k,0}(\tau_{k-1}(F^*)F^*) = (-1)^{k-1}/k! \quad (10.5)$$

When $k = s_j$, the generalized Divisor Axiom (cf. Lemma 1.4 of [KM]) and (10.4) (applied to $GW_{k,0}(\tau_{k-1}(F^*)F^*F^*)$) together with the facts $\phi_1 = 0$ on $\overline{\mathcal{M}}_{0,3}$ and $GW_{1,0}(F^*F^*) = 1$ give

$$GW_{k,0}(\tau_{k-1}(F^*)F^*) = \frac{1}{k} GW_{k,0}(\tau_{k-1}(F^*)F^*F^*) = -\frac{1}{k} GW_{k-1,0}(\tau_{k-2}(F^*)F^*)$$

By induction, this shows (10.5) that completes the proof. \square

Step 2 : We first show a formula for relative invariants that is analogous to (10.2) and then give a proof of Lemma 10.1. Recall that for the forgetful map $\pi_\ell : \overline{\mathcal{M}}_{g,n+\ell} \rightarrow \overline{\mathcal{M}}_{g,n}$ that forgets the last ℓ marked points and for $1 \leq i \leq n$ we have

$$\pi_\ell^* \phi_i = \phi_i - \sum \delta_{\{i\} \cup I} \quad (10.6)$$

where the sum is over all $I \subset \{n+1, \dots, n+\ell\}$ with $I \neq \emptyset$. For simplicity, we will write π_ℓ simply as π when ℓ is even.

Lemma 10.3. *Let $n \geq 3$. Then, for $s_j \geq 1$ we have*

$$\begin{aligned} GW_{(1^d), (1^d), g} \left(\prod_{i=1}^n \phi_i^{s_i} \pi^* \phi_i^{t_i} (F^*) \right) &= GW_{(1^d), (1^d), g} \left(\prod_{i=1}^n \phi_i^{s_i - \delta_{ij}} \pi^* \phi_i^{t_i + \delta_{ij}} (F^*) \right) \\ &- \sum_{0 < k < d} \delta_{ks_j} (-1)^{k-1} k! \binom{d}{k}^2 GW_{(1^{d-k}), (1^{d-k}), g} \left(\pi^* \phi_j^{t_j} (F^*) \prod_{i \neq j} \phi_i^{s_i} \pi^* \phi_i^{t_i} (F^*) \right) \end{aligned}$$

Proof. Without loss of generality, we may assume $j = 1$. For the forgetful map $\pi = \pi_{2d}$, let $\delta_{\{1\} \cup I}$ be a class as in (10.6) and denote by $\overline{\mathcal{M}}(\delta_{\{1\} \cup I})$ the boundary stratum of $\overline{\mathcal{M}}_{g,n+2d}$ whose fundamental class is Poincaré dual to $\delta_{\{1\} \cup I}$. Then for $m = |I|$ there is a gluing map

$$\eta_I : \overline{\mathcal{M}}_{0,m+2} \times \overline{\mathcal{M}}_{g,n+2d-m} \rightarrow \overline{\mathcal{M}}_{g,n+2d}$$

whose image is $\overline{\mathcal{M}}(\delta_{\{1\} \cup I})$. This gluing map is obtained by identifying the second marked point of the first component with the first marked point of the second component. We have

$$\eta_I^*(\phi_1) = \phi_1 \otimes 1 \quad \text{and} \quad \eta_I^* \circ \pi^*(\phi_1) = 1 \otimes \pi_{2d-m}^*(\phi_1) \quad (10.7)$$

where the first equality follows from Lemma 1.2 (b) and the second from the fact that under the composition map $\pi \circ \eta_I$ the first component collapses to a point.

Choose two distinct fibers V_1 and V_2 of \mathbb{P}_0 and, for simplicity, we set

$$\mathcal{M}^V = \mathcal{M}_{g,n,(1^d),(1^d)}^{V_1, V_2}(\mathbb{P}_0, dS) \quad \text{and} \quad \Phi = \phi_1^{s_1-1} \pi^* \phi_1^{t_1} \prod_{i>1} \phi_i^{s_i} \pi^* \phi_i^{t_i}$$

where $V = V_1 \sqcup V_2$. Let G be a geometric representative of the Poincaré dual of the pull-back class $\eta_I^* \Phi$. One can then choose a (smooth) family of geometric representatives G_t of the

Poincaré dual of the class $\delta_{\{1\} \cup I} \cup \Phi$ with $G_0 = \eta_I(G)$. Let B be a product of n distinct generic fibers B_i of \mathbb{P}_0 each of which is disjoint with V .

Suppose $\mathcal{M}^V \cap B \cap G_t \neq \emptyset$ for all small t . Then, by the Gromov Compactness Theorem, after passing to subsequences, as $t \rightarrow 0$ every sequence $f_t \in \mathcal{M}^V \cap B \cap G_t$ converges to

$$(f, C) \in C\mathcal{M}^V \cap B \cap \eta_I(G)$$

where $C\mathcal{M}^V$ is the closure of \mathcal{M}^V in $\overline{\mathcal{M}}_{g,n+2d}(\mathbb{P}_0, dS)$. The closure $C\mathcal{M}^V$ has a stratification in which each stratum consisting of maps with domains with more than one node has dimension at least 4 less than $2\deg(\Phi) + 2 + 2n$ (cf. Lemma 7.6 of [IP2]). Thus the domain $C \in \overline{\mathcal{M}}(\delta_I)$ of f has one node by dimension count. The limit map f splits as $f = (f_1, f_2)$ such that each f_i is V -regular unless it represents the trivial homology class. In our case, both f_1 and f_2 are V -regular maps since the image of f_1 passes through V and B_1 , and the image of f_2 passes through $(n-1) > 2$ distinct fibers B_i where $2 \leq i \leq n-1$. For some $0 < k < d$, we have

- f_1 (resp. f_2) has contact vector (1^k) (resp. (1^{d-k})) with both V_1 and V_2 , and hence
- $f \in ev_I^{-1}(\Delta) \cap B \cap \eta(G)$ (under the natural inclusion $ev_I^{-1}(\Delta) \hookrightarrow \mathcal{M}^V$)

where Δ is the diagonal of $\mathbb{P}_0 \times \mathbb{P}_0$ and ev_I is the evaluation map

$$ev_I = ev_2 \times ev_1 : \mathcal{M}_{0,2,(1^k),(1^k)}^{V_1,V_2}(\mathbb{P}_0, kS) \times \mathcal{M}_{g,n,(1^{d-k}),(1^{d-k})}^{V_1,V_2}(\mathbb{P}_0, (d-k)S) \mapsto \mathbb{P}_0 \times \mathbb{P}_0.$$

On the other hand, the condition of contact order (1^d) with V is an open condition. The Gluing Theorem of [RT2] thus implies that for each small t one can uniquely smooth f (at a node) to produce a V -regular map in $\mathcal{M}^V \cap G_t$. Consequently, we have

$$[ev_I^{-1}(\Delta)] \cap \eta_I^*(\Phi) \otimes (F^*)^n = [\mathcal{M}^V] \cap (\delta_{\{1\} \cup I} \cup \Phi) \otimes (F^*)^n. \quad (10.8)$$

It follows from (5.6), (10.7), (10.8), the splitting of the diagonal Δ as in the proof of Lemma 10.2 (see paragraph above (10.4)) that

$$\begin{aligned} & GW_{(1^d),(1^d),g} \left(\prod_{i=1}^n \phi_i^{s_i} \pi^* \phi_i^{t_i} (F^*) \right) = GW_{(1^d),(1^d),g} \left(\prod_{i=1}^n \phi_i^{s_i - \delta_{i1}} \pi^* \phi_i^{t_i + \delta_{i1}} (F^*) \right) \\ & - \sum_{0 < k < d} \binom{d}{k}^2 GW_{(1^k),(1^k),0}(\phi_1^{s_1-1}(F^*)F^*) \cdot GW_{(1^{d-k}),(1^{d-k}),g}(\pi^* \phi_1^{t_1}(F^*) \prod_{i>1} \phi_i^{s_i} \pi^* \phi_i^{t_i}(F^*)) \end{aligned} \quad (10.9)$$

where the factor $\binom{d}{k}^2$ reflects the fact that the classes $\delta_{\{1\} \cup I}$ in (10.8) are obtained by choosing k contact points with V_1 and k contact points with V_2 . Observe that by dimension if $k \neq s_1$ then $GW_{(1^k),(1^k),0}(\phi_1^{s_1-1}(F^*)F^*) = 0$. Thus, it remains to show :

$$GW_{(1^k),(1^k),0}(\phi^{k-1}(F^*)F^*) = (-1)^{k-1} k! \quad (10.10)$$

The sum formula (4.15) for $h = 0$ gives

$$\begin{aligned} GW_{(1^k),(1^k),0}(\phi^{k-1}(F^*)F^*F^*) &= \frac{1}{k!} GW_{(1^k),(1^k),0}(\phi^{k-1}(F^*)F^*) \cdot GT_{(1^k),(1^k)}^{\mathbb{P}_0}(F^*) \\ &= k GW_{(1^k),(1^k),0}(\phi^{k-1}(F^*)F^*) \end{aligned} \quad (10.11)$$

where the second equality follows from (3.8) and the Divisor Axiom. On the other hand, together with the facts $\phi = 0$ on $\overline{\mathcal{M}}_{0,3}$ and $GW_{(1),(1),0}(F^*F^*) = 1$, the formula (10.9) shows

$$GW_{(1^k),(1^k),0}(\phi^{k-1}(F^*)F^*F^*) = -k^2 GW_{(1^{k-1}),(1^{k-1}),0}(\phi^{k-2}(F^*)F^*). \quad (10.12)$$

By induction, (10.11) and (10.12) thus imply (10.10). This completes the proof. \square

Proof of Lemma 10.1 : It suffice to show that for $n \geq 3$

$$GW_{d,g}\left(\prod_{i=1}^n \tau_{s_i} \phi_i^{t_i}(F^*)\right) = \frac{1}{(d!)^2} GW_{(1^d),(1^d),g}\left(\prod_{i=1}^n \phi_i^{s_i} \pi^* \phi_i^{t_i}(F^*)\right). \quad (10.13)$$

When $\sum s_i = 0$, (10.13) follows from Lemma 5.2. Suppose that (10.13) holds for any d, g and $n \geq 3$ whenever $\sum s_i < \ell$. Then, Lemma 10.2 and Lemma 10.3 show that (10.13) also holds when $\sum s_i = \ell$. Therefore, (10.13) follows from induction on the sum $\sum s_i$. \square

References

- [ACV] D. Abramovich, A. Corti, and A. Vistoli, *Twisted Bundles and Admissible Covers*, Commun. in Algebra. **31** (2003), 3547-3618.
- [AC] E. Arbarello and M. Cornalba, *Calculating cohomology groups of moduli spaces of curves via algebraic geometry*, Inst. Hautes tudes Sci. Publ. Math. No. **88** (1998), 97-127.
- [C] M. Cornalba, *Moduli of curves and theta charateristics*, Lectures on Riemann Surfaces, 560-589, World Scientific, Singapore 1989.
- [FP] C. Faber and R. Pandharipande, *Hodge integrals and Gromov-Witten theory*, Invent. Math. **139** (2000), no. 1, 173-199.
- [HM] J. Harris and I. Morrison, *Moduli of curves*, Graduate Texts in Mathematics, 187. Springer-Verlag, New York, 1998.
- [IP1] E. Ionel and T.H. Parker, *Relative Gromov-Witten Invariants*, Annals of Math. **157** (2003), 45-96.
- [IP2] E. Ionel and T.H. Parker, *The Symplectic Sum Formula for Gromov-Witten Invariants*, Annals of Math. **159** (2004), 935-1025.
- [KL1] Y-H. Kiem and J. Li, *Gromov-Witten Invariants of Varieties with Holomorphic 2-Forms*, preprint, math.AG/07072986
- [KL2] Y-H. Kiem and J. Li, *Low degree GW invariants of spin surfaces*, Pure Appl. Math. Q. **7** (2011), no. 4, 1449-1476.
- [KL3] Y-H. Kiem and J. Li, *Low degree GW invariants of surfaces II*, Science China Math. **54** (2011), no. 8., 1679-1706.
- [KM] M. Kontsevich and Y.I. Manin, *Relations between the correlators of the topological sigma model coupled to gravity*, Commun. Math. Phys. **196** (1998), 385-398.

- [L] J. Lee, *Family Gromov-Witten Invariants for Kähler Surfaces*, Duke Math. J. **123** (2004), no. 1, 209–233.
- [Lo] E. Looijenga, *Smooth Deligne-Mumford compactifications by means of Prym level structures*, J. Algebraic Geom. **3** (1994), no. 2, 283–293.
- [LP1] J. Lee and T.H. Parker, *A Structure Theorem for the Gromov-Witten Invariants of Kähler Surfaces*, J. Diff. Geom. **77** (2007), no. 3, 483–513.
- [LP2] J. Lee and T.H. Parker, *An Obstruction bundle relating Gromov-Witten invariants of curves and Kähler surfaces*, Am. J. Math. **134** (2012), no. 2, 453–506.
- [LT] J. Li and G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of general symplectic manifolds*, Topics in symplectic 4-manifolds (Irvine, CA, 1996), 47–83, First Int. Press Lect. Ser., I, Internat. Press, Cambridge, MA, 1998.
- [MP] D. Maulik and R. Pandharipande, *New calculations in Gromov-Witten theory*, Pure Appl. Math. Q. **4** (2008), no. 2, part 1, 469–500.
- [RT1] Y. Ruan and G. Tian, *A mathematical theory of quantum cohomology*, J. Differential Geom. **42** (1995), 259–367.
- [RT2] Y. Ruan and G. Tian, *Higher genus symplectic invariants and sigma models coupled with gravity*, Invent. Math. **130** (1997), no. 3, 455–516.
- [T] L. Tu, *Hodge theory and the local Torelli problem*, Mem. Amer. Math. Soc. **43** (1983), no. 279.